544

A TREATISE ON

THE ANALYTIC GEOMETRY OF

THREE DIMENSIONS

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ANALYTIC GEOMETRY

OF

THREE DIMENSIONS

BY

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EDITOR'S PREFACE TO FIFTH EDITION.

VOLUME II.

In revising this volume I have been fortunate to have the assistance of Mr. G. R. Webb, Fellow of Trinity College, Dublin, Miss Hilda P. Hudson, Sc.D. (Dublin), of Newnham College, Cambridge, and Mr. Robert Russell, Fellow of Trinity College, Dublin. The two volumes now form, it is hoped, a concise and comprehensive survey of tri-dimensional Euclidean Geometry, both algebraic and differential, and those who wish to specialise further will find ample references to guide them.

Practically nothing has been omitted from the fourth edition, but the numbering of the articles has been altered in some places. New matter is enclosed in square brackets and the principal additions will now be described.

Chapter XIII. of the fourth edition has been divided into three Chapters, XIII., XIII*a* and XIII*b*; XIII*a* dealing with rectilinear complexes, rectilinear congruences and ruled surfaces, contains a good deal of new matter under the first two headings. The linear complex is treated analytically (Art. 454*a*), Arts. 455*a-o*, which were written by Mr. Russell, deal with the singular elements of complexes of any order

and, in particular, with the quadratic complex and Kummer's quartic surface. I have added some twenty-two pages on the differential properties of rectilinear congruences using Kummer's parametric method (455*q*-457*f*). The surfaces and points associated with a congruence are defined. Special attention is given to normal congruences with their optical and mechanical characteristics, and the investigation of those normal congruences which are defined by two directing curves (457*c*), leads naturally to a discussion of Dupin's cyclides (457*d*) which are the orthogonal surfaces. Arts. 457*e* and *f* contain a summary of Ribaucour's theorems dealing with the unique and interesting class of congruences known as isotropic or "circular," having the characteristic that the lines of striction of all ruled surfaces of the congruence lie on a surface. In 462*a* it is shown how to apply the parametric method to ruled surfaces.

A few examples of triply orthogonal systems have been added (486) with a brief account of Lamé's curvilinear co-ordinates, Lamé's equations and the connection between those and Cayley's differential equation (486*a*). In 486*b* and *c*, the Dupin-Darboux theorem has been proved and generalised so as to apply to "complexes" of curves. The subject naturally leads to an inquiry into normal congruences of curves. I have explained the parametric method of dealing with these, and as an illustration, I have given Ribaucour's beautiful theorem on the deformation of such congruences (486*d*). Cyclic systems are touched on in Arts. 486*e* and *f*; 495*a* and 515*a* are also new.

Chapters XV. and XVI. dealing with cubic and quartic surfaces have been revised and enlarged by Mr. G. R. Webb who has made several interesting additions. In 522*a*, *b*, *c*, further details are given on singular points of cubic surfaces with some account of Segre's method of analysing higher singularities; 527*a* contains a geometrical proof of the uniqueness of the canonical form of a cubic. In 527*b* the reader is introduced to the focal surface of the congruence of lines joining corresponding lines on the Hessian, this being the analogue of the Cayleyan of a plane cubic. In 536*a* various proofs are given of Schläfli's theorem, independently of the theory of cubic surfaces, and in 537 the connection is exhibited between the twenty-seven right lines on a cubic surface and the twenty-eight bitangents of a plane quartic.

In Chapter XVI. Mr. Webb has added a full bibliography of work done on the quartic surface (Art. 545), with articles on Steiner's quartic (554*a*), on the relation between Kummer's cones and the sixteen right lines on a quartic with nodal conic (559*a*), on cyclides with nodes (567*a*), on Weddle's surface, symmetroids, and Kummer's quartic surface (572*a* to 573*c*), and on Rohn's investigation of the maximum number of ovals that a quartic can possess (574). There are also some minor additions.

Chapter XVII. of the Fourth Edition has been sub-divided into two Chapters, XVII. and XVII*a*. This portion has been revised by Miss Hilda P. Hudson, who has also added six articles on the interesting subject of Cremona transformations

(Arts. 587*a-f*). The sections on the contact of lines and planes with surfaces (Arts. 588-608) have been re-arranged in a more logical order with minor additions. The principal changes are in Cayley's "addition on the theory of reciprocal surfaces" (Arts. 620-630). One is sorry to lose anything so picturesque as the "points of an unexplained singularity," but the phrase is no longer justified. This section has been re-written with very kind assistance from Professor Zeuthen to whom Miss Hudson and the editor offer their grateful acknowledgments.

REGINALD A. P. ROGERS.

TRINITY COLLEGE, DUBLIN,
October, 1914.

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422. LET the equations of a curve

$$\phi\ (x, y, z, c_1, c_2 \ldots c_n) = 0,\ \psi\ (x, y, z, c_1, c_2 \ldots c_n) = 0,$$

include n parameters, or undetermined constants; then it is evident that if n equations connecting these parameters be given, the curve is completely determined. If, however, only $n-1$ relations between the parameters be given, the equations above written may denote an infinity of curves; and the assemblage of all these curves constitutes a surface whose equation is obtained by eliminating the n parameters from the given $n+1$ equations; viz. the $n-1$ relations, and the two equations of the curve. Thus, for example, if the two equations above written denote a variable curve, the motion of which is regulated by the conditions that it shall intersect $n-1$ fixed directing curves, the problem is of the kind now under consideration. For, by eliminating x, y, z between the two equations of the variable curve, and the two equations of any one of the directing curves, we express the condition that these two curves should intersect, and thus have one relation between the n parameters. And having $n-1$ such relations we find the equation of the surface generated in the manner just stated. We had (Art. 112) a particular case of this problem.

Those surfaces for which the form of the functions ϕ and ψ is the same are said to be *of the same family*, though the equations connecting the parameters may be different. Thus, if the motion of the same variable curve were regulated by

several different sets of directing curves, all the surfaces generated would be said to belong to the same family. In several important cases, the equations of all surfaces belonging to the same family can be included in one equation involving one or more arbitrary functions, the equation of any individual surface of the family being then got by particularising the form of the functions. If we eliminate the arbitrary functions by differentiation, we get a partial differential equation, common to all surfaces of the family, which ordinarily is the expression of some geometrical property common to all surfaces of the family, and which leads more directly than the functional equation to the solution of some classes of problems.

423. The simplest case is *when the equations of the variable curve include but two constants.** Solving in turn for each of these constants, we can throw the two given equations into the form $u = c_1$, $v = c_2$; where u and v are known functions of x, y, z. In order that this curve may generate a surface, we must be given one relation connecting c_1, c_2, which will be of the form $c_1 = \phi\,(c_2)$; whence putting for c_1 and c_2 their values, we see that, whatever be the equation of connection, the equation of the surface generated must be of the form $u = \phi\,(v)$.

We can also, in this case, readily obtain the partial differential equation which must be satisfied by all surfaces of the family. For if $U = 0$ represents any such surface, U can only differ by a constant multiplier from $u - \phi\,(v)$. Hence, we have $\lambda U = u - \phi\,(v)$, and differentiating

$$\lambda U_1 = u_1 - \phi'\,(v)\, v_1,$$

with two similar equations for the differentials with respect to y and z. Eliminating then λ and $\phi'\,(v)$, we get the required partial differential equation in the form of a determinant.

* If there were but one constant, the elimination of it would give the equation of a definite surface, not of a family of surfaces.

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix} = 0.$$

In this case u and v are supposed to be known functions of the coordinates; and the equation just written establishes a relation of the first degree between U_1, U_2, U_3.

If the equation of the surface were written in the form $z - \phi\ (x, y) = 0$, we should have $U_3 = 1$, $U_1 = -p$, $U_2 = -q$, where p and q have the usual signification, and the partial differential equation of the family is of the form $Pp + Qq = R$, where P, Q, R are known functions of the coordinates. And, conversely, the integral of such a partial differential equation, which * is of the form $u = \phi\ (v)$, geometrically represents a surface which can be generated by the motion of a curve whose equations are of the form $u = c_1$, $v = c_2$.

The partial differential equation affords the readiest test whether a given surface belongs to any assigned family. We have only to give to U_1, U_2, U_3 their values derived from the equation of the given surface, which values must identically satisfy the partial differential equation of the family if the surface belong to that family.

424. If it be required to determine a particular surface of a given family $u = \phi\ (v)$, by the condition that the surface shall pass through a given curve, the form of the function in this case can be found by writing down the equations $u = c_1$, $v = c_2$, and eliminating x, y, z between these equations and those of the fixed curve; we thus find a relation between c_1 and c_2, or between u and v, which is the equation of the required surface. The geometrical interpretation of this process is, that we direct the motion of a variable curve $u = c_1$, $v = c_2$, by the condition that it shall move so as always to intersect the given fixed curve. All the points of the latter are therefore points on the surface generated.

If it be required to find a surface of the family $u = \phi\ (v)$

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* Boole's *Differential Equations*, p. 323, or Forsyth's *Differential Equations*, Art. 185.

which shall envelope a given surface, we know that at every point of the curve of contact U_1, U_2, U_3 have the same value for the fixed surface, and for that which envelopes it. If then, in the partial differential equation of the given family, we substitute for U_1, U_2, U_3 their values derived from the equation of the fixed surface, we get an equation which will be satisfied for every point of the curve of contact, and which therefore, combined with the equation of the fixed surface, determines that curve. The problem is, therefore, reduced to that considered in the first part of this article; namely, to describe a surface of the given family through a given curve. All this theory will be better understood from the following examples of important families of surfaces belonging to the class here considered; viz. whose equations can be expressed in the form $u = \phi(v)$.

425. *Cylindrical Surfaces.* A cylindrical surface is generated by the motion of a right line, which remains always parallel to itself. Now the equations of a right line include four independent constants; if then the direction of the right line be given, this determines two of the constants, and there remain but two undetermined. The family of cylindrical surfaces belongs to the class considered in the last two articles.

Thus, if the equations of a right line be given in the form $x = lz + p$, $y = mz + q$; l and m which determine the direction of the right line are supposed to be given; and if the motion of the right line be regulated by any condition (such as that it shall move along a certain fixed curve, or envelope a certain fixed surface) this establishes a relation between p and q, and the equation of the surface comes out in the form

$$x - lz = \phi(y - mz).$$

More generally, if the right line is to be parallel to the intersection of the two planes $ax + by + cz$, $a'x + b'y + c'z$, its equations must be of the form

$$ax + by + cz = \alpha, \quad a'x + b'y + c'z = \beta,$$

and the equation of the surface generated must be of the form

$$ax + by + cz = \phi(a'x + b'y + c'z).$$

Writing $ax+by+cz$ for u, and $a'x+b'y+c'z$ for v in the equation of Art. 423, we see that the partial differential equation of cylindrical surfaces is

$$(bc'-cb')\,U_1+(ca'-ac')\,U_2+(ab'-ba')\,U_3=0,$$

or (Ex. 3, Art. 44) $U_1\cos\alpha+U_2\cos\beta+U_3\cos\gamma=0$, where α, β, γ are the direction-angles of the generating line. Remembering that U_1, U_2, U_3 are proportional to the direction-cosines of the normal to the surface, it is obvious that the geometrical meaning of this equation is, that the tangent plane to the surface is always parallel to the direction of the generating line.

Ex. 1. To find the equation of the cylinder whose edges are parallel to $x=lz$, $y=mz$, and which passes through the plane curve $z=0$, $\phi(x, y)=0$.

Ans. $\phi(x-lz, y-mz)=0$.

Ex. 2. To find the equation of the cylinder whose sides are parallel to the intersection of $ax+by+cz$, $a'x+b'y+c'z$, and which passes through the intersection of $\alpha x+\beta y+\gamma z=\delta$, $F(x, y, z)=0$. Solve for x, y, z between the equations $ax+by+cz=u$, $a'x+b'y+c'z=v$, $\alpha x+\beta y+\gamma z=\delta$, and substitute the resulting values in $F(x, y, z)=0$.

Ex. 3. To find the equation of a cylinder, the direction-cosines of whose edges are l, m, n, and which passes through the curve $U=0$, $V=0$. The elimination may be conveniently performed as follows: If x', y', z' be the coordinates of the point where any edge meets the directing curve, x, y, z those of any point on the edge, we have $\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}$. Calling the common value of these functions θ, we have

$$x'=x-l\theta,\ y'=y-m\theta,\ z'=z-n\theta.$$

Substitute these values in the equations $U=0$, $V=0$, which $x'y'z'$ must satisfy, and between the two resulting equations eliminate the unknown θ and the result will be the equation of the cylinder.

Ex. 4. To find the cylinder, the direction-cosines of whose edges are l, m, n, and which envelopes the quadric $Ax^2+By^2+Cz^2=1$. From the partial differential equation, the curve of contact is the intersection of the quadric with

$$Alx+Bmy+Cnz=0.$$

Proceeding then, as in the last example, the equation of the cylinder is found to be

$$(Al^2+Bm^2+Cn^2)(Ax^2+By^2+Cz^2-1)=(Alx+Bmy+Cnz)^2.$$

426. *Conical Surfaces or Cones.* These are generated by the motion of a right line which constantly passes through a fixed point. Expressing that the coordinates of this point satisfy the equations of the right line, we have two relations

connecting the four constants in the general equations of a right line. In this case, therefore, the equations of the generating curve contain but two undetermined constants, and the problem is of the kind discussed, Art. 423.

Let the equations of the generating line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

where α, β, γ are the known coordinates of the vertex of the cone, and l, m, n are proportional to the direction-cosines of the generating line; and where the equations, though apparently containing three undetermined constants, actually contain only two, since we are only concerned with the ratios of the quantities l, m, n.

Writing the equations then in the form

$$\frac{x-\alpha}{z-\gamma}=\frac{l}{n}, \quad \frac{y-\beta}{z-\gamma}=\frac{m}{n},$$

we see that the conditions of the problem must establish a relation between $l:n$ and $m:n$, and that the equation of the cone must be of the form $\frac{x-\alpha}{z-\gamma}=\phi\left(\frac{y-\beta}{z-\gamma}\right)$.

It is easy to see that this is equivalent to saying that the equation of the cone must be a homogeneous function of the three quantities $x-\alpha$, $y-\beta$, $z-\gamma$; as may also be seen directly from the consideration that the conditions of the problem must establish a relation between the direction-cosines of the generator; that these cosines being $l : \sqrt{\{(l^2+m^2+n^2)\}}$, &c., any equation expressing such a relation is a homogeneous function of l, m, n, and therefore of $x-\alpha$, $y-\beta$, $z-\gamma$, which are proportional to l, m, n.

When the vertex of the cone is the origin, its equation is of the form $\frac{x}{z}=\phi\left(\frac{y}{z}\right)$; or, in other words, is a homogeneous function of x, y, z.

The partial differential equation is found by putting $u=\frac{x-\alpha}{z-\gamma}$, $v=\frac{y-\beta}{z-\gamma}$, in the equation of Art. 423, and when cleared of fractions is

$$\begin{vmatrix} U_1, & U_2, & U_3, \\ z-\gamma, & 0, & -(x-\alpha) \\ 0, & z-\gamma, & -(y-\beta) \end{vmatrix} = 0,$$

or $$(x-\alpha)\,U_1 + (y-\beta)\,U_2 + (z-\gamma)\,U_3 = 0.$$

This equation evidently expresses that the tangent plane at any point of the surface must always pass through the fixed point $\alpha\beta\gamma$.

We have already given in Ex. 9, Art. 121, the method of forming the equation of the cone standing on a given curve; and (Art. 277) the method of forming the equation of the cone which envelopes a given surface.

427. *Conoidal Surfaces.* These are generated by the motion of a line which always intersects a fixed axis and remains parallel to a fixed plane. These two conditions leave two of the constants in the equations of the line undetermined, so that these surfaces are of the class considered (Art. 423). If the axis is the intersection of the planes α, β, and the generator is to be parallel to the plane γ, the equations of the generator are $\alpha = c_1\beta$, $\gamma = c_2$, and the general equation of conoidal surfaces is obviously $\dfrac{\alpha}{\beta} = \phi\,(\gamma)$.

The partial differential equation is (Art. 423)

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ \beta a_1 - \alpha b_1, & \beta a_2 - \alpha b_2, & \beta a_3 - \alpha b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0,$$

where $\alpha = a_1 x + a_2 y + a_3 z + a_4$, &c. The left-hand side of the equation may be expressed as the difference of two determinants $\beta\,(U_1 a_2 c_3) - \alpha\,(U_1 b_2 c_3) = 0$.

This equation may be derived directly by expressing that the tangent plane at any point on the surface contains the generator; the tangent plane, therefore, the plane drawn through the point on the surface, parallel to the directing plane, and the plane $\alpha'\beta - \alpha\beta'$ joining the same point to the axis, have a common line of intersection. The terms of the determinant just written are the coefficients of x, y, z in the equations of these three planes.

In practice we are almost exclusively concerned with *right conoids;* that is, where the fixed axis is perpendicular to the directing plane. If that axis be taken as the axis of z, and the plane for plane of xy, the functional equation is $y = x\phi(z)$, and the partial differential equation is $xU_1 + yU_2 = 0$.

The lines of greatest slope (Art. 421) are in this case always orthogonally projected into circles on the directing plane. For in virtue of the partial differential equation just written, the equation of Art. 421,

$$U_2dx - U_1dy = 0,$$

transforms itself into $xdx + ydy = 0$, which represents a series of concentric circles. The same thing is evident geometrically; for the lines of level are the generators of the system; and these being projected into a series of radii all passing through the origin, are cut orthogonally by a series of concentric circles.

Ex. 1. To find the equation of the right conoid passing through the axis of z and through a plane curve, whose equations are $x = a$, $F(y, z) = 0$. Eliminating then x, y, z between these equations and $y = c_1x$, $z = c_2$, we get $F(c_1a, c_2) = 0$; or the required equation is $F\left(\frac{ay}{x}, z\right) = 0$.

Wallis's cono-cuneus is when the fixed curve is a circle $x = a$, $y^2 + z^2 = r^2$. Its equation is therefore $a^2y^2 + x^2z^2 = r^2x^2$.

Ex. 2. Let the directing curve be a helix, the fixed line being the axis of the cylinder on which the helix is traced. The equation is that given Ex. 1, Art. 371. This surface is often presented to the eye, being that formed by the under surface of a spiral staircase.

[Ex. 3. A right conoid of special interest in Statics and Dynamics is Ball's Cylindroid.* If θ be the angle between the generator and a fixed plane through the axis, the surface is defined by the relation

$$z = h \sin 2\theta$$

and its equation is therefore

$$z(x^2 + y^2) - 2hxy = 0.$$

Ex. 4. Any right conoid may be expressed by two parameters by equations of the form

$$x = p\cos q, \quad y = p\sin q, \quad z = \phi(q).$$

Thus for the conoid of Ex. 2, $\phi(q) \equiv \frac{q}{n}$. (See Ex. 1, Art. 371.)]

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* R. S. Ball, *The Theory of Screws* (Dublin, 1876).

Ex. 5. The equation of any surface generated by the motion of a right line meeting two fixed right lines $\alpha\beta$, $\gamma\delta$, must be of the form $\frac{\alpha}{\beta} = \phi\left(\frac{\gamma}{\delta}\right)$.

428. *Surfaces of Revolution.* The fundamental property of a surface of revolution is that its section perpendicular to its axis must always consist of one or more circles whose centres are on the axis. Such a surface may therefore be conceived as generated by a circle of variable radius whose centre moves along a fixed right line or axis, and whose plane is perpendicular to that axis. If the equations of the axis be $\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, then the generating circle in any position may be represented as the intersection of the plane perpendicular to the axis $lx + my + nz = c_1$, with the sphere whose centre is any fixed point on the axis,

$$(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = c_2.$$

These equations contain but two undetermined constants; the problem, therefore, is of the class considered (Art. 423), and the equation of the surface must be of the form

$$(x-a)^2 + (y-\beta)^2 + (z-\gamma)^2 = \phi\,(lx + my + nz).$$

When the axis of z is the axis of revolution, we may take the origin as the point $a\beta\gamma$, and the equation becomes

$$x^2 + y^2 + z^2 = \phi\,(z), \text{ or } z = \psi\,(x^2 + y^2).$$

The partial differential equation is found by the formula of Art. 423 to be

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ l, & m, & n \\ x-a, & y-\beta, & z-\gamma \end{vmatrix} = 0,$$

or $\{m\,(z-\gamma) - n\,(y-\beta)\}\,U_1$

$$+ \{n\,(x-a) - l\,(z-\gamma)\}\,U_2 + \{l\,(y-\beta) - m\,(x-a)\}\,U_3 = 0.$$

When the axis of z is the axis of revolution, this reduces to

$$yU_1 - xU_2 = 0.$$

The partial differential equation expresses that the normal always meets the axis of revolution. For, if we wish to express the condition that the two lines

$$\frac{x-a}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \quad \frac{x-x'}{U_1} = \frac{y-y'}{U_2} = \frac{z-z'}{U_3},$$

should intersect, we may write the common value of the equal fractions in each case, θ and θ'. Solving then for x, y, z, and equating the values derived from the equations of each line, we have

$$\alpha + l\theta = x' + U_1\theta', \beta + m\theta = y' + U_2\theta', \gamma + n\theta = z' + U_3\theta';$$

whence, eliminating θ, θ', the result is the determinant already found

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ l, & m, & n \\ x' - \alpha, & y' - \beta, & z' - \gamma \end{vmatrix} = 0.$$

[Ex. 1. Any surface of revolution may be expressed by two parameters as follows:

$$x = p \cos q, \quad y = p \sin q, \quad z = \phi(p).$$

Ex. 2. By a suitable choice of parameters (u, v) the square of the linear element (Art. 377) of a surface of revolution may be expressed in the form

$$ds^2 = f(u)(du^2 + dv^2)$$

and hence (Art. 390), *any surface whose linear element can be thus expressed is deformable into a surface of revolution.*

For example, the right conoid of Ex. 2, Art. 427, satisfies this condition as may be seen by using the form given in Ex. 4 of the same Art.]

429. The equation of the surface generated by the revolution of a given curve round a given axis is found (Art. 424) by eliminating x, y, z between

$$lx + my + nz = u, \quad (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = v,$$

and the two equations of the curve; replacing then u and v by their values. We have already had an example of this (Ex. 3, Art. 121), and we take, as a further example, *to find the surface generated by the revolution of a circle*

$$y = 0, \quad (x - a)^2 + z^2 = r^2$$

round an axis in its plane (the axis of z).

Putting $z = u$, $x^2 + y^2 = v$, and eliminating between these equations and those of the circle, we get

$$\{\sqrt{(v)} - a\}^2 + u^2 = r^2, \quad \text{or} \quad \{\sqrt{(x^2 + y^2)} - a\}^2 + z^2 = r^2,$$

which, cleared of radicals, is

$$(x^2 + y^2 + z^2 + a^2 - r^2)^2 = 4a^2(x^2 + y^2).$$

It is obvious that when a is greater than r, that is to say, when the revolving circle does not meet the axis, neither can the surface, which will be the form of an *anchor ring*, the space

about the axis being empty. On the other hand, when the revolving circle meets the axis, the segments into which the axis divides the circle generate distinct sheets of the surface, intersecting in points on the axis $z = \sqrt{(r^2 - a^2)}$, which are nodal points on the surface.

The sections of the anchor ring by planes parallel to the axis are found by putting $y =$ constant, in the preceding equation. The equation of the section may immediately be thrown into the form $SS' =$ constant, where S and S' represent circles. The sections are Cassinians of various kinds (see fig. *Higher Plane Curves*, p. 44). It is geometrically evident, that as the plane of section moves away from the axis, it continues to cut in two distinct ovals, until it touches the surface $y = a - r$ when it cuts in a curve having a double point (Bernoulli's Lemniscate); after which it cuts in a continuous curve.

Ex. Verify that $x^3 + y^3 + z^3 - 3xyz = r^3$ is a surface of revolution.

Ans. The axis of revolution is $x = y = z$.

430. The families of surfaces which have been considered are the most interesting of those whose equations can be expressed in the form $u = \phi\,(v)$. We now proceed to the *case when the equations of the generating curve include more than two parameters.* By the help of the equations connecting these parameters, we can, in terms of any one of them, express all the rest, and thus put the equations of the generating curve into the form

$$F\{x, y, z, c, \phi\,(c), \psi\,(c), \text{\&c.}\} = 0, f\{x, y, z, c, \phi\,(c), \psi\,(c), \text{\&c.}\} = 0.$$

The equation of the surface generated is obtained by eliminating c between these equations; and, as has been already stated, all surfaces are said to be of the same family for which the form of the functions F and f is the same, whatever be the forms of the functions ϕ, ψ, &c. But since evidently the elimination cannot be effected until some definite form has been assigned to the functions ϕ, ψ, &c., it is not generally possible to form a single functional equation including all surfaces of the same family; and we can only represent them, as above written, by a pair of equations from which there

remains a constant to be eliminated. We can, however, eliminate the arbitrary functions by differentiation, and obtain a partial differential equation, common to all surfaces of the same family; the order of that equation being, as we shall presently prove, equal to the number of arbitrary functions ϕ, ψ, &c.

It is to be remarked, however, that in general the order of the partial differential equation obtained by the elimination of a number of arbitrary functions from an equation is higher than the number of functions eliminated. Thus, if an equation include two arbitrary functions ϕ, ψ, and if we differentiate with respect to x and y, which we take as independent variables, the differential equations combined with the original one form a system of three equations containing four unknown functions ϕ, ψ, ϕ', ψ'. The second differentiation (twice with regard to x, twice with regard to y, and with regard to x and y) gives us three additional equations; but, then, from the system of six equations it is not generally possible to eliminate the six quantities ϕ, ψ, ϕ', ψ', ϕ'', ψ''. We must, therefore, proceed to a third differentiation before the elimination can be effected. It is easy to see, in like manner, that to eliminate n arbitrary functions we must differentiate $2n-1$ times. The reason why, in the present case, the order of the differential equation is less, is that the functions eliminated are all functions *of the same quantity*.

431. In order to show this, it is convenient to consider first the special case, where a family of surfaces can be expressed by a single functional equation. This will happen when it is possible by combining the equations of the generating curve to separate one of the constants so as to throw the equations into the form

$$u = c_1,\ F(x, y, z, c_1, c_2 \ldots c_n) = 0.$$

Then expressing, by means of the equations of condition, the other constants in terms of c_1, the result of elimination is plainly of the form

$$F\{x, y, z, u, \phi(u), \psi(u), \&c.\} = 0.$$

Now, if we denote by F_1, the differential with respect to x of the equation of the surface, on the supposition that u is constant, and similar differentials in y, z by F_2, F_3, we have

$$U_1 = F_1 + \frac{dF}{du} u_1, \quad U_2 = F_2 + \frac{dF}{du} u_2, \quad U_3 = F_3 + \frac{dF}{du} u_3.$$

But, in these equations, the derived functions ϕ', ψ', &c., only enter in the term $\frac{dF}{du}$; they can, therefore, be all eliminated together, and we can form the equation, homogeneous in U_1, U_2, U_3,

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ F_1, & F_2, & F_3 \\ u_1, & u_2, & u_3 \end{vmatrix} = 0,$$

which contains only the original functions ϕ, ψ, &c. If we write this equation $V = 0$, we can form from it, in like manner, the equation

$$\begin{vmatrix} U_1, & U_2, & U_3 \\ V_1, & V_2, & V_3 \\ u_1, & u_2, & u_3 \end{vmatrix} = 0,$$

which still contains no arbitrary functions but the original ϕ, ψ, &c., but which contains the second differential coefficients of U, these entering into V_1, V_2, V_3. From the equation last found we can in like manner form another, and so on; and from the series of equations thus obtained (the last being of the n^{th} order of differentiation) we can eliminate the n functions ϕ, ψ, &c.

If we omit the last of these equations we can eliminate all but one of the arbitrary functions, and according to our choice of the function to be retained, can obtain n different equations of the order $n-1$, each containing one arbitrary function. These are the first integrals of the final differential equation of the n^{th} order. In like manner we can form $\frac{1}{2}n\,(n-1)$ equations of the second order, each containing two arbitrary functions, and so on.

432. If we take x and y as the independent variables, and as usual write $dz = pdx + qdy$, $dp = rdx + sdy$, &c., the process

of forming these equations may be more conveniently stated as follows: "Take the total differential of the given equation on the supposition that u is constant,

$$F_1dx + F_2dy + F_3\ (pdx + qdy) = 0\ ;$$

put $dy = mdx$, and substitute for m its value derived from the differential of $u = 0$, viz.

$$u_1dx + u_2dy + u_3\ (pdx + qdy) = 0."$$

For, if we differentiate the given equation with respect to x and y, we get

$$F_1 + pF_3 + \frac{dF}{du}\ (u_1 + pu_3) = 0,$$

$$F_2 + qF_3 + \frac{dF}{du}\ (u_2 + qu_3) = 0,$$

and the result of eliminating $\frac{dF}{du}$ from these two equations is the same as the result of eliminating m between the equations

$$F_1 + pF_3 + m\ (F_2 + qF_3) = 0,\ u_1 + pu_3 + m\ (u_2 + qu_3) = 0.$$

It is convenient in practice to choose for one of the equations representing the generating curve its projection on the plane of xy; then, since this equation does not contain z, the value of m derived from it will not contain p or q, and the first differential equation will be of the form

$$p + qm = R,$$

R being also a function not containing p or q. The only terms then containing r, s, or t in the second differential equation are those derived from differentiating $p + qm$, and that equation will be of the form

$$r + 2sm + tm^2 = S,$$

where S may contain x, y, z, p, q, but not r, s, or t. If now we had only two functions to eliminate, we should solve for these functions from the original functional equation of the surface, and from $p + qm = R$; and then substituting these values in m and in S, the *form* of the final second differential equation would still remain

$$r + 2sm' + tm'^2 = S',$$

where m' and S' might contain x, y, z, p, q. In like manner if we had three functions to eliminate, and if we denote the

partial differentials of z of the third order by α, β, γ, δ, the partial differential equation would be of the form

$$\alpha + 3m\beta + 3m^2\gamma + m^3\delta = T.$$

And so on for higher orders. This theory will be illustrated by the examples which follow.

433. *Surfaces generated by lines parallel to a fixed plane.* This is a family of surfaces which includes conoids as a particular case. Let us, in the first place, take the fixed plane for the plane of xy. Then the equations of the generating line are of the form $z = c_1$, $y = c_2x + c_3$. The functional equation of the surface is got by substituting in the latter equation for c_2, $\phi(z)$, and for c_3, $\psi(z)$. Since in forming the partial differential equation we are to regard z as constant, we may as well leave the equations in the form $z = c_1$, $y = c_2x + c_3$. These give us

$$p + qm = 0,\ m = c_2.$$

According as we eliminate c_3 or c_2, these equations give us $p + qc_2 = 0$, $px + qy = qc_3$. There are, therefore, two equations of the first order, each containing one arbitrary function, viz.

$$p + q\phi(z) = 0,\ px + qy = q\psi(z).$$

To eliminate arbitrary functions completely, differentiate $p + qm = 0$, remembering that since $m = c_2$, it is to be regarded as constant, when we get

$$r + 2sm + tm^2 = 0,$$

and eliminating m by means of $p + qm = 0$, the required equation is

$$q^2r - 2pqs + p^2t = 0,$$

Next let the generating line be parallel to $ax + by + cz$; its equations are

$$ax + by + cz = c_1, \quad y = c_2x + c_3;$$

and the functional equation of the family of surfaces is got by writing for c_2 and c_3, functions of $ax + by + cz$. Differentiating, we have

$$a + cp + m(b + cq) = 0, \quad m = c_2.$$

The equations got by eliminating one arbitrary function are therefore

$$a + cp + (b + cq)\,\phi\,(ax + by + cz) = 0,$$
$$(a + cp)\,x + (b + cq)\,y = (b + cq)\,\psi\,(ax + by + cz).$$

Differentiating $a + bm + c\,(p + mq) = 0$, and remembering that m is to be regarded as constant, we have

$$r + 2sm + tm^2 = 0,$$

and introducing the value of m already found,

$$(b + cq)^2\,r - 2\,(a + cp)\,(b + cq)\,s + (a + cp)^2 t = 0.$$

434. This equation may also be arrived at by expressing that the tangent planes at two points on the same generator intersect, as they evidently must, on that generator. Let α, β, γ be the running coordinates, x, y, z those of the point of contact; then any generator is the intersection of the tangent plane

$$\gamma - z = p\,(\alpha - x) + q\,(\beta - y),$$

with a plane through the point of contact parallel to the fixed plane

$$a\,(\alpha - x) + b\,(\beta - y) + c\,(\gamma - z) = 0,$$

whence $\quad (a + cp)\,(\alpha - x) + (b + cq)\,(\beta - y) = 0.$

Now if we pass to the line of intersection of this tangent plane with a consecutive plane, α, β, γ remain the same, while x, y, z, p, q vary. Differentiating the equation of the tangent plane, we have

$$(rdx + sdy)\,(\alpha - x) + (sdx + tdy)\,(\beta - y) = 0.$$

And eliminating $\alpha - x$, $\beta - y$,

$$(b + cq)\,(rdx + sdy) = (a + cp)\,(sdx + tdy).$$

But since the point of contact moves along the generator which is parallel to the fixed plane, we have

$$adx + bdy + cdz = 0, \text{ or } (a + cp)\,dx + (b + cq)\,dy = 0.$$

Eliminating then dx, dy from the last equation, we have, as before,

$$(b + cq)^2\,r - 2\,(a + cp)\,(b + cq)\,s + (a + cp)^2\,t = 0.$$

435. *Surfaces generated by lines which meet a fixed axis.* This class also includes the family of conoids. In the first place let the fixed axis be the axis of z; then the equations of the generating line are of the form $y = c_1 x$, $z = c_2 x + c_3$; and

the equation of the family of surfaces is got by writing in the latter equation for c_2 and c_3, arbitrary functions of $y : x$. Differentiating, we have $m = c_1$, $p + mq = c_2$, whence

$$px + qy = x\phi\left(\frac{y}{x}\right), \text{ and } z - px - qy = \psi\left(\frac{y}{x}\right).$$

Differentiating again, we have $r + 2sm + tm^2 = 0$, and putting for m its value $= c_1 = \frac{y}{x}$, the required differential equation is

$$rx^2 + 2sxy + ty^2 = 0.$$

This equation may also be obtained by expressing that two consecutive tangent planes intersect in a generator. As in the last article, we have for the intersection of two consecutive tangent planes

$$(rdx + sdy)(\alpha - x) + (sdx + tdy)(\beta - y) = 0.$$

But any generator lies in the plane

$$\alpha y = \beta x, \text{ or } (\alpha - x)\, y = (\beta - y)\, x.$$

Eliminating therefore,

$$x\,(rdx + sdy) + y\,(sdx + tdy) = 0.$$

But $\frac{dy}{dx} = \frac{\beta}{\alpha} = \frac{y}{x}$. Therefore, as before, $rx^2 + 2sxy + ty^2 = 0$.

More generally, let the line pass through a fixed axis $\alpha\beta$, where $\alpha = ax + by + cz + d$, $\beta = a'x + b'y + c'z + d'$. Then the equations of the generating line are $\alpha = c_1\beta$, $y = c_2 x + c_3$, and the equation of the family of surfaces is $y = x\phi\left(\frac{\alpha}{\beta}\right) + \psi\left(\frac{\alpha}{\beta}\right)$.

Differentiating, we have

$$m = c_2, \quad a + cp + m\,(b + cq) = c_1\,\{a' + c'p + m\,(b' + c'q)\}.$$

Differentiating again, we have $r + 2sm + tm^2 = 0$, and putting in for m from the last equation, the required partial differential equation is

$$\{(a + cp)\,\beta - (a' + c'p)\alpha\}^2\, t + \{(b + cq)\,\beta - (b' + c'q)\alpha\}^2\, r$$
$$- 2\,\{(a + cp)\,\beta - (a' + c'p)\alpha\}\,\{(b + cq)\,\beta - (b' + c'q)\alpha\}\, s = 0.$$

436. If the equation of a family of surfaces contain n arbitrary functions of the same quantity, and if it be required *to determine a surface of the family which shall pass through n fixed curves,* we write down the equations of the generating

curve $u=c_1$, $F(x, y, z, c_1, c_2, \&c.)=0$, and expressing that the generating curve meets each of the fixed curves, we have a sufficient number of equations to eliminate c_1, c_2, &c. Thus, to find a surface of the family $x+y\phi(z)+\psi(z)=0$ which shall pass through the fixed curves

$$y=a, \quad F(x, z)=0; \quad y=-a, \quad F_1(x, z)=0.$$

The equations of the generating line being $z=c_1$, $x=yc_2+c_3$, we have, by substitution,

$$F(ac_2+c_3, c_1)=0, \quad F_1(c_3-ac_2, c_1)=0,$$

or, replacing for c_1, c_3, their values,

$$F\{x+c_2(a-y), z\}=0, \quad F_1\{x-c_2(a+y), z\}=0,$$

and by eliminating c_2 between these the required surface is found.

Ex. Let the directing curves be

$$y=a, \quad \frac{x^2}{b^2}+\frac{z^2}{c^2}=1; \quad y=-a, \quad x^2+z^2=c^2.$$

We eliminate c_2 between

$$\frac{\{x+c_2(a-y)\}^2}{b^2}+\frac{z^2}{c^2}=1, \quad \{x-c_2(a+y)\}^2+z^2=c^2.$$

Solving for c_2 from each, we have

$$\frac{\frac{b}{c}\sqrt{(c^2-z^2)}-x}{a-y}=\frac{x-\sqrt{(c^2-z^2)}}{a+y}.$$

The result is apparently of the eighth degree, but is resolvable into two conoids distinguished by giving the radicals the same or opposite signs in the last equation.

437. We have now seen, that when the equation of a family of surfaces contains a number of arbitrary functions of the same quantity, it is convenient, in forming the partial differential equation, to substitute for the equation of the surface, the two equations of the generating curve. It is easy to see, then, that this process is equally applicable when the family of surfaces cannot be expressed by a single functional equation. The arbitrary functions which enter into the equations (Art. 430) are all functions of the same quantity, though the expression of that quantity in terms of the coordinates is unknown. If then differentiating that quantity gives $dy=mdx$, we can eliminate the unknown quantity m,

between the total differentials of the two equations of the generating curve, and so obtain the partial differential equation required. In practice it is convenient to choose for one of the equations of the generating curve, its projection on the plane xy.

For example, let it be required to find the *general equation of ruled surfaces*: that is to say, of surfaces generated by the motion of the right line. The equations of the generating line are

$$z = c_1 x + c_3, \ y = c_2 x + c_4,$$

and the family of surfaces is expressed by substituting for c_2, c_3, c_4 arbitrary functions of c_1. Differentiating, we have

$$p + mq = c_1, \ m = c_2.$$

Differentiating the first of these equations, m being proved to be constant by the second, we have

$$r + 2sm + tm^2 = 0.$$

As this equation still includes m or c_2, the expression for which, in terms of the coordinates is unknown, we must differentiate again, when we have

$$\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = 0,$$

where α, β, γ, δ are the third differential coefficients. Eliminating m between the cubic and quadratic just found, we have the required partial differential equation. It evidently resolves itself into the two linear equations of the third order got by substituting in turn for m in the cubic the two roots of the quadratic.

This equation might be got geometrically by expressing that the tangent planes at three consecutive points on a generator pass through that generator. The equation

$$pdx + qdy = dz$$

is a relation between p, q, -1, which are proportional to the direction-cosines of a tangent plane, while dx, dy, dz are proportional to the direction-cosines of any line in that plane passing through the point of contact. If, then, we pass to a second tangent plane, through a consecutive point on the same line, we are to make p, q vary while the mutual ratios of dx, dy, dz remain constant. This gives

$$rdx^2 + 2sdxdy + tdy^2 = 0.$$

To pass to a third tangent plane, we differentiate again, regarding $dx : dy$ constant; and thus have

$$adx^3 + 3\beta dx^2dy + 3\gamma dxdy^2 + \delta dy^3 = 0.$$

Eliminating $dx : dy$ between the last two equations, we have the same equation as before.

The first integrals of this equation are found, as explained (Art. 431), by omitting the last equation and eliminating all but one of the constants. Thus we have the equation $p + mq = c_1$, from which it appears that one of the integrals is

$$p + mq = \phi(m),$$

where m is one of the roots of $r + 2sm + tm^2 = 0$. The other two first integrals are

$$y - mx = \psi(m), \quad \text{and} \quad z - px - mqx = \chi(m).$$

The three second integrals are got by eliminating m from any pair of these equations.

438. *Envelopes.* If the equation of a surface include n parameters connected by $n - 1$ relations, we can in terms of any one express all the rest, and throw the equation into the form

$$z = F\{x, y, c, \phi(c), \psi(c), \&c.\}.$$

Eliminating c between this equation and $\dfrac{dF}{dc} = 0$, which we shall write $F' = 0$, we find the envelope of all the surfaces obtained by giving different values to c. The envelopes so found are said to be of the same family as long as the form of the function F remains the same, no matter how the forms of the functions ϕ, ψ, &c., vary. The curve of intersection of the given surface with F' is the *characteristic* (see p. 30) or line of intersection of two consecutive surfaces of the system. Considering the characteristic as a moveable curve from the two equations of which c is to be eliminated, it is evident that the problem of envelopes is included in that discussed Art. 430, &c. If the function F contain n arbitrary functions ϕ, ψ, &c., then since F' contains ϕ', ψ', &c., it would seem, according to the theory previously explained, that the partial differential equation of

the family ought to be of the $2n^{\text{th}}$ order. But on examining the manner in which these functions enter, it is easy to see that the order reduces to the n^{th}. In fact, differentiating the equation $z=F$, we get

$$p=F_1+\frac{dF}{dc}c_1,\ q=F_2+\frac{dF}{dc}c_2, \text{ that is, } p=F_1+c_1F',\ q=F_2+c_2F',$$

but since $F'=0$, we have $p=F_1$, $q=F_2$, where, since F_1 and F_2 are the differentials on the supposition that c is constant, these quantities only contain the original functions ϕ, ψ and not the derived ϕ', ψ'. From this pair of equations we can form another, as in the last article, and so on, until we come to the n^{th} order, when, as easily appears from what follows, we have equations enough to eliminate all the parameters.

439. We need not consider the case when the given equation contains but one parameter, since the elimination of this between the equation and its differential gives rise to the equation of a definite surface and not of a family of surfaces. *Let the equation then contain two parameters a, b, connected by an equation giving b as a function of a*, then between the three equations $z=F$, $p=F_1$, $q=F_2$, we can eliminate a, b, and the form of the result is evidently $f(x, y, z, p, q)=0$.

For example, let us examine *the envelope of a sphere of fixed radius, whose centre moves along any plane curve in the plane of xy*. This is a particular case of the general class of tubular surfaces which we shall consider presently.

Now the equation of such a sphere being

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2,$$

and the conditions of the problem assigning a locus along which the point $\alpha\beta$ is to move, and therefore determining β in terms of α, the equation of the envelope is got by eliminating α between

$$(x-\alpha)^2+\{y-\phi(\alpha)\}^2+z^2=r^2,\ (x-\alpha)+\{y-\phi(\alpha)\}\,\phi'(\alpha)=0.$$

Since the elimination cannot be effected until the form of the function ϕ is assigned, the family of surfaces can only be expressed by the combination of two equations just written. We might also obtain these equations by expressing that the

surface is generated by a fixed circle, which moves so that its plane shall be always perpendicular to the path along which its centre moves. For the equation of the tangent to the locus of $\alpha\beta$ is

$$y-\beta=\frac{d\beta}{d\alpha}(x-\alpha) \quad \text{or} \quad y-\phi(\alpha)=\phi'(\alpha)\;(x-\alpha).$$

And the plane perpendicular to this is

$$(x-\alpha)+\{y-\phi(\alpha)\}\,\phi'(\alpha)=0,$$

as already obtained. To obtain the partial differential equation, differentiate the equation of the sphere, regarding α, β as constant, when we have

$$x-\alpha+pz=0,\; y-\beta+qz=0.$$

Solving for $x-\alpha$, $y-\beta$ and substituting in the equation of the sphere, the required equation is

$$z^2(1+p^2+q^2)=r^2.$$

We might have at once obtained this equation as the geometrical expression of the fact that the length of the normal is constant and equal to r, as it obviously is.

440. Before proceeding further we wish to show how the arbitrary functions which occur in the equation of a family of envelopes can be determined by the conditions that the surface in question passes through given curves. The tangent line to one of the given curves at any point of course lies in the tangent plane to the required surface; but since the enveloping surface has at any point the same tangent plane as the enveloped surface which passes through that point, it follows that each of the given curves at every point of it touches the enveloped surface which passes through that point. If, then, the equation of the enveloped surface be

$$z=F(x,\,y,\,c_1,\,c_2\ldots c_n),$$

the envelope of this surface can be made to pass through $n-1$ given curves; for by expressing that the surface, whose equation has just been written, touches each of the given curves, we obtain $n-1$ relations between the constants c_1, c_2, &c., which, combined with the two equations of the characteristic, enable us to eliminate these constants.

For example, the family of surfaces discussed in the last article contains but two constants and one arbitrary function, and can therefore be made to pass through one given curve. Let it then be required to find an envelope of the sphere

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2,$$

which shall pass through the right line $x=mz$, $y=0$. The points of intersection of this line with the sphere being given by the quadratic

$$(mz-\alpha)^2+\beta^2+z^2=r^2, \text{ or } (1+m^2)z^2-2mz\alpha+\alpha^2+\beta^2-r^2=0,$$

the condition that the line should touch the sphere is

$$(1+m^2)(\alpha^2+\beta^2-r^2)=m^2\alpha^2.$$

We see thus, that the locus of the centres of spheres touching the given line is an ellipse. The envelope required, then, is a kind of elliptical anchor ring, whose equation is got by eliminating α, β between

$$(x-\alpha)^2+(y-\beta)^2+z^2=r^2,\ (1+m^2)\ (\alpha^2+\beta^2-r^2)=m^2\alpha^2,$$

$$(x-\alpha)\,d\alpha+(y-\beta)\,d\beta=0,\ \alpha d\alpha+(1+m^2)\,\beta d\beta=0,$$

from which last two equations we have

$$(1+m^2)\beta\,(x-\alpha)=\alpha\,(y-\beta).$$

The result is a surface of the eighth degree.

441. Again, let it be required to determine the arbitrary function so that the enveloping surface may also envelope a given surface. At any point of contact of the required surface with the fixed surface $z=f(x, y)$, the moveable surface $z=F\,(x, y, c_1, c_2, \&c.)$ which passes through that point, has also the same tangent plane as the fixed surface. The values then of p and q derived from the equations of the fixed surface and of the moveable surface must be the same. Thus we have $f_1=F_1$, $f_2=F_2$, and if between these equations and the two equations $z=F$, $z=f$, which are satisfied for the point of contact, we eliminate x, y, z, the result will give a relation between the parameters. The envelope may thus be made to envelope as many fixed surfaces as there are arbitrary functions in the equation.

Thus, for example, let it be required to determine a tubular surface of the kind discussed in the last article, which shall

touch the sphere $x^2+y^2+z^2=R^2$. This surface must then touch $(x-a)^2+(y-\beta)^2+z^2=r^2$. We have therefore

$$x:y:z=x-a:y-\beta:z;$$

conditions which imply $z=0$, $\beta x=ay$. Eliminating x and y by the help of these equations, between the equation of the fixed and moveable sphere, we get

$$4(a^2+\beta^2)R^2=(R^2-r^2+a^2+\beta^2)^2.$$

This gives a quadratic for $a^2+\beta^2$, whose roots are $(R\pm r)^2$; showing that the centre of the moveable sphere moves on one or other of two circles, the radius being either $R+r$ or $R-r$. The surface required is therefore one or other of two anchor rings, the opening of the rings corresponding to the values just assigned.

442. We add one or two more examples of families of envelopes whose equations include but one arbitrary function. To find the envelope of a right cone whose axis is parallel to the axis of z, and whose vertex moves along any assigned curve in the plane of xy. Let the equation of the cone in its original position be $z^2=m^2(x^2+y^2)$; then if the vertex be moved to the point a, β, the equation of the cone becomes $z^2=m^2\{(x-a)^2+(y-\beta)^2\}$, and if we are given a curve along which the vertex moves, β is given in terms of a. Differentiating, we have $pz=m^2(x-a)$, $qz=m^2(y-\beta)$; and eliminating, we have

$$p^2+q^2=m^2.$$

This equation expresses that the tangent plane to the surface makes a constant angle with the plane of xy, as is evident from the mode of generation. It can easily be deduced hence, that the area of any portion of the surface is in a constant ratio to its projection on the plane of xy.

443. The families of surfaces, considered (Arts. 439, 442), are both included in the following: *To find the envelope of a surface of any form which moves without rotation, its motion being directed by a curve along which any given point of the surface moves.* Let the equation of the surface in its original

position be $z = F(x, y)$, then if it be moved without turning so that the point originally at the origin shall pass to the position $\alpha\beta\gamma$, the equation of the surface will evidently be $z - \gamma = F(x - \alpha, y - \beta)$. If we are given a curve along which the point $\alpha\beta\gamma$ is to move, we can express α, β in terms of γ, and the problem is one of the class to be considered in the next article, where the equation of the envelope includes two arbitrary functions. Let it be given, however, that the directing curve *is drawn on a certain known surface*, then, of the two equations of the directing curve, one is known and only one arbitrary, so that the equation of the envelope includes but one arbitrary function. Thus, if we assume β an arbitrary function of α, the equation of the fixed surface gives γ as a known function of α, β. It is easy to see how to find the partial differential equation in this case. Between the three equations

$$z - \gamma = F(x - \alpha, y - \beta), p = F_1(x - \alpha, y - \beta), q = F_2(x - \alpha, y - \beta),$$

solve for $x - \alpha$, $y - \beta$, $z - \gamma$, when we find

$$x - \alpha = f(p, q), \quad y - \beta = f'(p, q), \quad z - \gamma = f''(p, q).$$

If, then, the equation of the surface along which $\alpha\beta\gamma$ is to move be $\Gamma(\alpha, \beta, \gamma) = 0$, the required partial differential equation is

$$\Gamma\{x - f(p, q), y - f'(p, q), z - f''(p, q)\} = 0.$$

The three functions f, f', f'', are evidently connected by the relation $df'' = pdf + qdf'$.

It is easy to see that the partial differential equation just found is the expression of the fact, that the tangent plane at any point on the envelope is parallel to that at the corresponding point on the original surface.

Ex. To find the partial differential equation of the envelope of a sphere of constant radius whose centre moves along any curve traced on a fixed equal sphere

$$x^2 + y^2 + z^2 = r^2.$$

The equation of the moveable sphere is $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$, whence

$$x - \alpha + p(z - \gamma) = 0, \quad y - \beta + q(z - \gamma) = 0,$$

and we have

$$x - \alpha = \frac{-pr}{(1 + p^2 + q^2)^{\frac{1}{2}}}, \quad y - \beta = \frac{-qr}{(1 + p^2 + q^2)^{\frac{1}{2}}}, \quad z - \gamma = \frac{r}{(1 + p^2 + q^2)^{\frac{1}{2}}}.$$

If we write $1 + p^2 + q^2 = \rho^2$ it is easy to see, by actual differentiation, that the relation is fulfilled

$$d\frac{1}{\rho} = -pd\left(\frac{p}{\rho}\right) - qd\left(\frac{q}{\rho}\right).$$

The partial differential equation is

$$(x\rho + pr)^2 + (y\rho + qr)^2 + (z\rho - r)^2 = \rho^2 r^2,$$

or $$(x^2 + y^2 + z^2)(1 + p^2 + q^2)^{\frac{1}{2}} + 2(px + qy - z)r = 0.$$

444. We now proceed to investigate the form of *the partial differential equation of the envelope, when the equation of the moveable surface contains three constants connected by two relations.* If the equation of the surface be $z = F(x, y, a, b, c)$, then we have $p = F_1$, $q = F_2$. Differentiating again, as in Art. 432, we have

$$r + sm = F_{11} + mF_{12}, \quad s + tm = F_{12} + mF_{22};$$

and eliminating m, the required equation * is

$$(r - F_{11})(t - F_{22}) = (s - F_{12})^2.$$

The functions F_{11}, F_{12}, F_{22} contain a, b, c, for which we are to substitute their values in terms of p, q, x, y, z derived from solving the preceding three equations, when we obtain an equation of the form

$$Rr + 2Ss + Tt + U(rt - s^2) = V,$$

where R, S, T, U, V are connected by the relation

$$RT + UV = S^2.$$

445. The following examples are among the most important of the cases where the equation includes three parameters.

Developable Surfaces. These are the envelope of the plane $z = ax + by + c$, where for b and c we may write $\phi(a)$ and $\psi(a)$. Differentiating, we have $p = a$, $q = b$, whence $q = \phi(p)$. *Any surface therefore is a developable surface if p and q are connected by a relation independent of x, y, z.* Thus the family (Art. 442) for which $p^2 + q^2 = m^2$, is a family of developable surfaces. We have also $z - px - qy = \psi(p)$, which is the

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* I owe to Professor Boole my knowledge of the fact, that when the equation of the moveable surface contains three parameters, the partial differential equation is of the form stated above. See his Memoir, *Phil. Trans.*, 1862, p. 437.

other first integral of the final differential equation. This last is got by differentiating again the equations $p=a$, $q=b$, when we have $r+sm=0$, $s+tm=0$, and eliminating m,

$$rt-s^2=0,$$

which is the required equation.

By comparing Arts. 285, 295, 311, it appears that the condition $rt=s^2$ is satisfied at every parabolic point on a surface. The same thing may be shown directly by transforming the equation $rt-s^2=0$ into a function of the differential coefficients of U, by the help of the relations

$$U_1+pU_3=0,\quad U_2+qU_3=0,$$

$$U_{11}+2U_{13}p+U_{33}p^2=-rU_3;\; U_{12}+pU_{23}+qU_{13}+pqU_{33}=-sU_3;$$

$$U_{22}+2U_{23}q+U_{33}q^2=-tU_3;$$

when the equation $rt-s^2=0$ is found to be identical with the equation of the Hessian. We see, accordingly, that every point on a developable is a parabolic point, as is otherwise evident, for since (Art. 330) the tangent plane at any point meets the surface in two coincident right lines, the two inflexional tangents at that point coincide. *The Hessian of a developable must therefore always contain the equation of the surface itself as a factor.* The Hessian of a surface of any degree n being of the degree $4n-8$, that of a developable consists of the surface itself, and a surface of $3n-8$ degree which we shall call the *Pro-Hessian.*

In order to find in what points the developable is met by the Pro-Hessian, I form the Hessian of the developable surface of the r^{th} degree (see Arts. 329, 330) $xu+y^2v=0$, and find that we get the developable itself multiplied by a series of terms in which the part independent of x and y is $v\left\{\dfrac{d^2u}{dz^2}\dfrac{d^2u}{dw^2}-\left(\dfrac{d^2u}{dwzd}\right)^2\right\}$. This proves that any generator xy meets the Pro-Hessian in the first place, where xy meets v; that is to say, twice in the point on the cuspidal curve (m), and in $r-4$ points on the nodal curve (x), Art. 330; and in the second place, where the generator meets the Hessian of u considered as a binary quantic; that is to say, in the Hessian of the system formed by these $r-4$ points combined

with the point on (m) taken three times; in which Hessian the latter point will be included four times. The intersection of any generator with the Pro-Hessian consists of the point on (m) taken six times, of the $r-4$ points on (x), and of $2(r-5)$ other points, in all $3r-8$ points.*

446. *Tubular Surfaces.* Let it be required to find the differential equation of *the envelope of a sphere of constant radius, whose centre moves on any curve.* We have, as in Art. 443,

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=R^2,$$
$$x-\alpha+p(z-\gamma)=0,\quad y-\beta+q(z-\gamma)=0,$$

whence
$$1+p^2+(z-\gamma)r+m\{pq+(z-\gamma)s\}=0,$$
$$pq+(z-\gamma)s+m\{1+q^2+(z-\gamma)t\}=0.$$

And therefore
$$\{1+p^2+(z-\gamma)r\}\{1+q^2+(z-\gamma)t\}=\{pq+(z-\gamma)s\}^2.$$

Substituting for $z-\gamma$ its value $\dfrac{R}{(1+p^2+q^2)^{\frac{1}{2}}}$ from the first three equations, this becomes

$$R^2(rt-s^2)-R\{(1+q^2)r-2pqs+(1+p^2)t\}\sqrt{(1+p^2+q^2)}$$
$$+(1+p^2+q^2)^2=0,$$

which denotes, Art. 311, that *at any point on the required envelope one of the two principal radii of curvature is equal to R,* as is geometrically evident.

447. We shall briefly show what the form of the differential equation is *when the equation of the surface whose envelope is sought contains four constants.* We have, as before, in addition to the equation of the surface, the three equations

$$p=F_1,\ q=F_2,\ (r-F_{11})(t-F_{22})=(s-F_{12})^2.$$

* Cayley has calculated the equation of the Pro-Hessian (*Quarterly Journal*, vol. VI p. 108) in the case of the developables of the fourth and fifth orders, and of that of the sixth order considered, Art. 348. The Pro-Hessian of the developable of the fourth order is identical with the developable itself. In the other two cases the cuspidal curve is a cuspidal curve also on the Pro-Hessian, and is counted six times in the intersection of the two surfaces. I suppose it may be assumed that this is generally true. The nodal curve is but a simple curve on the Pro-Hessian, and therefore is only counted twice in the intersection.

Let us, for shortness, write the last equation $\rho\tau = \sigma^2$, and let us write $\alpha - F_{111} = A$, $\beta - F_{112} = B$, $\gamma - F_{122} = C$, $\delta - F_{222} = D$; then, differentiating $\rho\tau = \sigma^2$, we have

$$(A + Bm)\,\tau + (C + Dm)\,\rho - 2\,(B + Cm)\,\sigma = 0.$$

Substituting for m from the equation $\sigma + \tau m = 0$, and remembering that $\rho\tau = \sigma^2$, we have

$$A\tau^3 - 3B\sigma\tau^2 + 3C\sigma^2\tau - D\sigma^3 = 0,$$

in which equation we are to substitute for the parameters implicitly involved in it, their values derived from the preceding equations. The equation is, therefore, of the form

$$\alpha + 3\beta m + 3\gamma m^2 + \delta m^3 = U,$$

where m and U are functions of x, y, z, p, q, r, s, t. In like manner we can form the differential equation when the equation of the moveable surface includes a greater number of parameters.

448. Having in the preceding articles explained how partial differential equations are formed, we shall next show how *from a given partial differential equation can be derived another differential equation satisfied by every characteristic of the family of surfaces to which the given equation belongs* (see Monge, p. 53). In the first place, let the given equation be of the *first order*; that is to say, of the form

$$f(x, y, z, p, q) = 0.$$

Now if this equation belong to the envelope of a moveable surface, it will be satisfied, not only by the envelope, but also by the moveable surface in any of its positions. This follows from the fact that the envelope touches the moveable surface, and therefore that at the point of contact x, y, z, p, q are the same for both. Now if x, y, z be the coordinates of any point on the characteristic, since such a point is the intersection of the two consecutive positions of the moveable surface, the equation $f(x, y, z, p, q) = 0$ will be satisfied by these values of x, y, z, whether p and q have the values derived from one position of the moveable surface or from the next consecutive. Consequently, if we differentiate the given

equation, regarding p and q as alone variable, then the points of the characteristic must satisfy the equation

$$Pdp + Qdq = 0.$$

Or we might have stated the matter as follows: Let the equation of the moveable surface be $z = F(x, y, a)$, where the constants have all been expressed as functions of a single parameter a. Then (Art. 438) we have $p = F_1(x, y, a)$, $q = F_2(x, y, a)$, which values of p and q may be substituted in the given equation. Now the characteristic is expressed by combining with the given equation its differential with respect to a; and a only enters into the given equation in consequence of its entering into the values for p and q. Hence we have, as before, $P\frac{dp}{da} + Q\frac{dq}{da} = 0$.

Now since the tangent line to the characteristic at any point of it lies in the tangent plane to either of the surfaces which intersect in that point, the equation $dz = pdx + qdy$ is satisfied, whether p and q have the values derived from one position of the moveable surface or from the next consecutive. We have therefore $\frac{dp}{da}dx + \frac{dq}{da}dy = 0$. And combining this equation with that previously found, we obtain the differential equation of the characteristic $Pdy - Qdx = 0$.

Thus, if the given equation be of the form $Pp + Qq = R$, the characteristic satisfies the equation $Pdy - Qdx = 0$, from which equation, combined with the given equation and with $dz = pdx + qdy$, can be deduced $Pdz = Rdx$, $Qdz = Rdy$. The reader is aware* of the use made of these equations in integrating this class of equations. In fact, if the above system of simultaneous equations integrated give $u = c_1$, $v = c_2$, these are the equations of the characteristic or generating curve in any of its positions, while in order that v may be constant whenever u is constant we must have $u = \phi(v)$.

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* See Boole's *Differential Equations*, p. 323, and Forsyth's *Differential Equations*, Art. 185.

Ex. Let the equation be that considered (Art. 439), viz. $z^2(1 + p^2 + q^2) = r^2$, then any characteristic satisfies the equation $pdy = qdx$, which indicates (Art. 421) that the characteristic is always a line of greatest slope on the surface, as is geometrically evident.

449. The equation just found for the characteristic generally includes p and q, but we can eliminate these quantities by combining with the equation just found the given partial differential equation and the equation $dz = pdx + qdy$. Thus, in the last example, from the equations $z^2(1+p^2+q^2) = r^2$, $qdx = pdy$, we derive

$$z^2(dx^2 + dy^2 + dz^2) = r^2(dx^2 + dy^2).$$

The reader is aware that there are two classes of differential equations of the first order, one derived from the equation of a single surface, as, for instance, by the elimination of any constant from an equation $U = 0$, and its differential

$$U_1dx + U_2dy + U_3dz = 0.$$

An equation of this class expresses a relation between the direction-cosines of every tangent line drawn at any point on the surface. The other class is obtained by combining the equations of two surfaces, as, for instance, by eliminating three constants between the equations $U = 0$, $V = 0$, and their differentials. An equation of this second class expresses a relation satisfied by the direction-cosines of the tangent to any of the curves which the system U, V represents for any value of the constants. The equations now under consideration belong to the latter class. Thus the geometrical meaning of the equation chosen for the example is, that the tangent to any of the curves denoted by it makes with the plane of xy an angle whose cosine is $z : r$. This property is true of every circle in a vertical plane whose radius is r; and the equation might be obtained by eliminating by differentiation the constants α, β, m, between the equations

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2,\ x - \alpha + m(y - \beta) = 0.$$

450. The differential equation found, as in the last article, is not only true for every characteristic of a family of surfaces, but since each characteristic touches the cuspidal edge of the

surface generated, the ratios $dx : dy : dz$ are the same for any characteristic and the corresponding cuspidal edge; and consequently the equation now found is satisfied by the cuspidal edge of every surface of the family under consideration. Thus, in the example chosen, the geometrical property expressed by the differential equation not only is true for a circle in a vertical plane, but remains true if the circle be wrapped on any vertical cylinder; and the cuspidal edge of the given family of surfaces always belongs to the family of curves thus generated.

Precisely as a partial differential equation in p, q (expressing as it does a relation between the direction-cosines of the tangent plane) is true as well for the envelope as for the particular surfaces enveloped, so the total differential equations here considered are true both for the cuspidal edge and the series of characteristics which that edge touches. The same thing may be stated otherwise as follows: the system of equations $U=0$, $\frac{dU}{da}=0$, which represents the characteristic when a is regarded as constant, represents the cuspidal edge when a is an unknown function of the variables to be eliminated by means of the equation $\frac{d^2U}{da^2}=0$. But the equations $U=0$, $\frac{dU}{da}=0$ evidently have the same differentials as if a were constant, when a is considered to vary, subject to this condition.

Thus, in the example of the last article, if in the equations $(x-a)^2+(y-\beta)^2+z^2=r^2$, $(x-a)+m(y-\beta)=0$, we write $\beta=\phi\ (a)$, $m=\phi'\ (a)$, and combine with these the equation $1+\phi'\ (a)^2=(y-\beta)\ \phi''\ (a)$, the differentials of the first and second equations are the same when a is variable, in virtue of the third equation, as if it were constant; and therefore the differential equation obtained by eliminating a, β, m between the first two equations and their differentials, on the supposition that these quantities are constant, holds equally when they vary according to the rules here laid down. And

we shall obtain the equations of a curve satisfying this differential equation by giving any form we please to $\phi(a)$, and then eliminating a between the equations

$$(x-a)^2+\{y-\phi(a)\}^2+z^2=r^2,\ (x-a)+\phi'(a)\{y-\phi(a)\}=0,$$
$$1+\{\phi'(a)\}^2=\{y-\phi(a)\}\phi''(a).$$

It is convenient to insert here a remark made by M. Roberts, viz. that if in the equation of any surface we substitute for x, $x+\lambda dx$, for y, $y+\lambda dy$, for z, $z+\lambda dz$, and then form the discriminant with respect to λ, the result will be the differential equation of the cuspidal edge of any developable enveloping the given surface. In fact it is evident (see Art. 277) that the discriminant expresses the condition that the tangent to the curve represented by it touch the given surface. Thus the general equation of the cuspidal edge of developables circumscribing a sphere is

$$(x^2+y^2+z^2-a^2)(dx^2+dy^2+dz^2)=(xdx+ydy+zdz)^2,$$

or
$$(ydz-zdy)^2+(zdx-xdz)^2+(xdy-ydx)^2=a^2(dx^2+dy^2+dz^2).$$

In the latter form it is evident that the same equation is satisfied by a geodesic traced on any cone whose vertex is the origin. For if the cone be developed into a plane, the geodesic will become a right line; and if the distance of that line from the origin be a, then the area of the triangle formed by joining any element ds to the origin is half ads, but this is evidently the property expressed by the preceding equation.

451. In like manner can be found the differential equation of the characteristic, the given partial differential equation being of the *second order* (see Monge, p. 74). In this case we can have two consecutive surfaces, satisfying the given differential equation, and touching each other all along their line of intersection. For instance, if we had a surface generated by a curve moving so as to meet two fixed directing curves, we might conceive a new surface generated by the same curve meeting two new directing curves, and if these latter directing curves touch the former at the points where the generating curve meets them, it is evident that the two surfaces touch along this line. In the case supposed, then, the two surfaces have x, y, z, p, q common along their line of intersection and can differ only with regard to r, s, t. Differentiate then the given differential equation, considering these quantities alone variable, and let the result be

$$Rdr+Sds+Tdt=0,$$

But, since p and q are constant along this line we have

$$drdx + dsdy = 0, \quad dsdx + dtdy = 0.$$

Eliminating then dr, ds, dt, the required equation for the characteristic is

$$Rdy^2 - Sdxdy + Tdx^2 = 0.$$

In the case of all the equations of the second order, which we have already considered, this equation turns out a perfect square. When it does not so turn out, it breaks up into two factors, which, if rational, belongs to two independent characteristics represented by separate equations; and if not, denote two branches of the same curve intersecting on the point of the surface which we are considering.

452. In fact, when the motion of a surface is regulated by a single parameter (see Art. 321), the equation of its envelope, as we have seen, contains only functions of a single quantity, and the differential equation belongs to the simpler species just referred to. But if the motion of the surface be regulated by two parameters, its contact with its envelope being not a curve, but a point, then the equation of the envelope will in general contain functions of two quantities, and the differential equation will be of the more general form. As an illustration of the occurrence of the latter class of equations in geometrical investigations, we take the *equation of the family of surfaces which has one set of its lines of curvature parallel to a fixed plane*, $y = mx$. Putting $dy = mdx$ in the equation of Art. 310, the differential equation of the family is

$$m^2 \{(1+q^2)\, s - pqt\} + m\, \{(1+q^2)\, r - (1+p^2)\, t\} \\ - \{(1+p^2)\, s - pqr\} = 0.$$

As it does not enter into the plan of this work to treat of the integration of such equations, we refer to Monge, p. 161, for a very interesting discussion of this equation. Our object being only to show how such differential equations present themselves in geometry, we shall show that the preceding equation arises from the elimination of a, β between the following equation and its differentials with respect to a and β:

$$(x-a)^2 + (y-\beta)^2 + \{z - \phi\, (a+m\beta)\}^2 = \{\psi\, (\beta - ma)\}^2.$$

Differentiating with respect to α and β, we have

$$(x-\alpha)+(z-\phi)\,\phi'=m\,\psi'\psi,$$
$$(y-\beta)+m\,(z-\phi)\,\phi'=-\psi'\psi,$$

whence $\quad (x-\alpha)+m\,(y-\beta)+(1+m^2)\,(z-\phi)\,\phi'=0.$

But we have also

$$(x-\alpha)+p\,(z-\phi)=0,\quad (y-\beta)+q\,(z-\phi)=0,$$

whence $\quad (x-\alpha)=m\,(y-\beta)+(p+mq)\,(z-\phi)=0.$

And, by comparison with the preceding equation, we have $p+mq=(1+m^2)\,\phi'\,(\alpha+m\beta)$. If, then, we call $\alpha+m\beta$, γ, the problem is reduced to eliminating γ between the equations

$$x+my-\gamma+(p+mq)\,\{z-\phi\,(\gamma)\}=0,\ p+mq=(1+m^2)\,\phi'\,(\gamma).$$

Differentiating with regard to x and y, we have

$$(1+p^2+mpq)+(r+ms)\,\{z-\phi\,(\gamma)\}=\{1+(p+mq)\,\phi'\}\,\gamma_1,$$
$$\{m\,(1+q^2)+pq\}+(s+mt)\,\{z-\phi\,(\gamma)\}=\{1+(p+mq)\,\phi'\}\gamma_2,$$

but from the second equation

$$r+ms:s+mt::\gamma_1:\gamma_2.$$

Hence, the result is

$$(1+p^2+mpq)\,(s+mt)=\{m\,(1+q^2)+pq\}\,(r+ms),$$

as was to be proved.

CHAPTER XIII (*a*).

COMPLEXES, CONGRUENCES, RULED SURFACES.*

453. THE preceding families of cylindrical surfaces, conical surfaces and conoidal surfaces, are all included in the more general family of ruled surfaces; but it is natural to consider these from a somewhat different point of view. We start with the right line, as a curve containing four parameters. Considering these as arbitrary, we have the whole system of lines in space; but we may imagine the parameters connected

* In W. R. Hamilton's second supplement on Systems of Rays, *Transactions of the Royal Irish Academy*, vol. XVI. 1830, were first investigated the properties of a congruence other than that formed by the normals to a surface. As to the theory of complexes and congruences see Plücker's posthumous work, *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement*, Leipzig, 1868, edited by Klein; also Kummer's Memoirs, *Crelle*, LVII. 1860, p. 189; and "Ueber die algebraischen Strahlensysteme, ins Besondere über die der ersten und zweiten Ordnung," *Berl. Abh.* 1866, pp. 1-120; and various Memoirs by Klein and others.

As regards ruled surfaces see Chasles's Memoir, Quetelet's *Correspondance*, t. XI. p. 50, and Cayley's paper, *Cambridge and Dublin Mathematical Journal*, vol. VII. p. 171; also his Memoir, "On Scrolls otherwise Skew Surfaces," *Philosophical Transactions*, 1863, p. 453, and later Memoirs. [See his *Collected Papers*. See also Plücker, *Theorie générale des surfaces réglées leur classification et leur construction*, Ann. di Mat. II. 1, 1867.]

[The literature of the subject is very extensive, and the reader who wishes for a more complete bibliography may refer to Loria, *Il Passato ed il Presente delle Principali Teorie Geometriche* (3rd ed. Turin, 1907), p. 207 *sqq.*, p. 407 *sqq.* (complexes and congruences), and p. 120 *sqq.*; p. 374 (ruled surfaces). We may refer especially to Jessop, *A Treatise on the Line-Complex* (Cambridge, 1903) for a compendious treatment of algebraic complexes; and to Bianchi, *Lezioni di Geometria Differenziale* (2nd ed. Pisa, 1902), or to Eisenhart's *Differential Geometry* (Boston, 1909), which is based on Bianchi's work, for summaries of the more interesting differential properties of rectilinear congruences. On isotropic congruences see Ribaucour's Memoir cited on p. 75.]

by a single equation, or by two, three, or four equations (more accurately, by a one-fold, two-fold, three-fold or four-fold relation). In the last case we have merely a system consisting of a finite number of right lines, and this may be excluded from consideration; the remaining cases are those of a one-fold, two-fold, and three-fold relation, or may be called those of a triple, double, or single system of right lines.

A. The parameters have a one-fold relation. We have here what Plücker has termed a *complex* of lines. As examples, we have the system of lines which touch any given surface whatever, or which meet any given curve whatever, but it is important to notice, as has been already remarked in Art. 80*d* and in Art. 316 (*D*), that these are particular cases only; the lines belonging to a complex do not in general touch one and the same surface, or meet one and the same curve.

We may, in regard to an algebraic complex, ask how many of the lines thereof meet each of three given lines, and the number in question may be regarded as the *order* of the complex.

B. The parameters have a two-fold relation. We have here a *congruence* of lines. A well-known example is that of the normals of a given surface. Each of these touches at two points (the centres of curvature) a certain surface, the centro-surface or locus of the centres of curvature of the given surface, and the normals are thus bitangents of the centro-surface. And so, in general, we have as a congruence of lines the system of the bitangents of a given surface. But more than this, *every congruence of lines may be regarded as the system of the bitangents of a certain surface*, for each line of the congruence is in general met by two consecutive lines, and the locus of the points of intersection is the surface in question.* The surface may, however, break up into two separate surfaces, and the original surface, or each or either of the component surfaces may degenerate into a curve; we have thus as congruences the systems of lines,

(1) the bitangents of a surface,

* See Art. 457.

(2) lines "through two points" of a curve,
(3) common tangents of two surfaces,
(4) tangents to a surface from the points of a curve,
(5) common transversals of two curves,

the last four cases being, as it were, degenerate cases of the first, which is the general one.

We may, in regard to [an algebraic] congruence, ask how many of the lines thereof meet each of two given lines? the number in question is the *order-class* of the congruence. But imagine the two given lines to intersect; the lines of the congruence are either the lines which pass through the point of intersection of the two given lines, or else the lines which lie in the common plane of the two given lines, and the questions thus arise: (1) How many of the lines of the congruence pass through a given point? the number is the *order* of the congruence. (2) How many of the lines of the congruence lie in a given plane? the number is the *class* of the congruence.

[A surface generated by the lines of a congruence which meet a given directing curve may be termed a *ruled surface of the congruence.* Let U be a ruled surface whose directing curve is a right line α. If a right line β intersects α in P, the lines of the congruence intersecting both α and β are those passing through P, and those lying in the plane containing α and β; that is, their number is $m+n$. Hence the degree of the surface U is in general $m+n$.

Hence also *the order-class of an algebraic congruence is equal to the sum of the order and class;* for the number of rays of the congruence meeting any two right lines α and γ is equal to the number $(m+n)$ of points in which the ruled surface U meets the line γ.]

C. Relation between the parameters three-fold. We have here a *regulus* of lines or ruled-surface, that generated by a series of lines depending on a single variable parameter. The *order* or *degree* of the system is the number of lines of the system which meet a given right line.

SECTION I.—RECTILINEAR COMPLEXES.

454. In accordance with Plücker's work on the right line considered as an element of space, we must therefore first consider the properties of a *rectilinear complex;* that is to say, of a system of lines which satisfy a single relation between the six coordinates. If this relation be of the n^{th} degree, the complex is of the n^{th} degree; all the lines of it which pass through a given point form a cone of the n^{th} order, and those which lie in a given plane envelope a curve of the n^{th} class (see Art. 80*d*). [The order is twice the degree.]

If, for instance, the complex be of the first degree, all the lines which pass through a given point lie in a given plane, the *polar plane* of the point; and, reciprocally, those which lie in a given plane pass through a given point, the *pole* of the given plane. To each line in space corresponds a conjugate or *polar* line, the points of the one line corresponding to the planes which pass through the other. Any line which meets two conjugate lines will be a line of the complex. When five lines of such a complex are given, it is evident, by counting the number of constants, that the complex is determined; and what has just been said enables us to construct geometrically the plane answering to any point. For, taking any four lines of the complex, the two lines which meet these four are conjugate lines, and the line passing through the assumed point and meeting the conjugate lines is a line of the complex. A second line is determined in like manner, and the two together determine the plane.

If we consider a series of parallel planes, to each corresponds a single point, and the locus of these points is therefore a line of the first degree, which right line may be called *the diameter* of the system of planes. To the plane infinity corresponds a point at infinity, and through this point all the diameters pass; that is to say, they are parallel.

One of the diameters is perpendicular to the corresponding plane, and this diameter may be called the *axis* of the complex.

If the axis and a line of the complex be given, the complex is determined. If a line of the complex be translated parallel to the axis or rotated round it, it still belongs to the complex.

When the line meets the axis we have the limiting case of a complex, the *special complex*, consisting of all lines which meet a given one, the *directrix*. It will be remembered (Art. 57*c*) that the condition that a complex shall be of this nature is that its coefficients shall satisfy the equation

$$PS + QT + RU = 0,$$

the equation of the complex being

$$Pp + Qq + Rr + Ss + Tt + Uu = 0.$$

[Ex. 1. The equation of a linear complex being

$$Pp + Qq + Rr + Ss + Tt + Uu = 0,$$

prove that the coordinates of the polar line of any line $(p, q, r, s, t, u,)$ are

$$p - \lambda S,\ q - \lambda T,\ r - \lambda U,\ s - \lambda P,\ t - \lambda Q,\ u - \lambda R,$$

where $$\lambda = \frac{Pp + Qq + Rr + Ss + Tt + Uu}{PS + QT + RU}.$$

Ex. 2. A line of the complex coincides with its polar line.

Ex. 3. The polar line of any line with regard to a *special* complex coincides with the directrix.]

[454*a*. *Simplest form of equation of linear complex. Geometrical construction.* The polar plane of any point $x'y'z'w'$ is

$$P(yz' - y'z) + Q(zx' - z'x) + R(xy' - x'y) + S(xw' - x'w)$$
$$+ T(yw' - y'w) + U(zw' - z'w) = 0.$$

And therefore, using rectangular Cartesian coordinates, the poles of the parallel planes whose equations for different values of μ are

$$\mu(ax + by + cz) + d = 0$$

where a, b, c, d are constants, lie on the right line

$$\frac{-Qz + Ry + S}{a} = \frac{Pz - Rx + T}{b} = \frac{-Py + Qx + U}{c}$$

which is the equation of the diameter of this system of planes. It is easy to see then that *the direction-cosines of all diameters are in the ratio $P : Q : R$.*

Since the direction-cosines of the perpendicular to the polar plane of any point xyz are proportional to

$$-Qz+Ry+S,\ Pz-Rx+T,\ -Py+Qx+U,$$

we find by expressing the condition that the diameter through a point is perpendicular to the polar plane, that the *equation of the axis of the complex is*

$$\frac{-Qz+Ry+S}{P}=\frac{Pz-Rx+T}{Q}=\frac{-Py+Qx+U}{R}.$$

The direction-cosines of the perpendicular to the polar plane of the origin are in the ratio $S:T:U$, and those of a diameter through the same point are in the ratio $P:Q:R$. Therefore if the axis of z be taken as the axis of the complex, we must have $P=0$, $Q=0$, $S=0$, $T=0$, *and the equation of the complex is reduced to the simple form*

$$R\,(xy'-yx')+U(z-z')=0, \text{ or}$$
$$xy'-yx'=h\,(z-z').$$

From this we can derive a *geometrical construction for the linear complex.* Let r be the shortest distance between the axis and a ray of the complex passing through xyz and $x'y'z'$, and let θ be the angle between the ray and the axis. Then

$$r=\frac{xy'-yx'}{\sqrt{(x-x')^2+(y-y')^2}}$$

and
$$\tan\theta=\frac{\sqrt{(x-x')^2+(y-y')^2}}{z-z'}.$$

Thus
$$xy'-yx'=(z-z')\ r\tan\theta$$
$$\text{and } h=r\tan\theta.$$

Thus the linear complex is defined by the property that *the shortest distance between a ray and the axis, multiplied by the tangent of the angle between the ray and the axis, is constant.*

If we put $x-x'=dx$, $y-y'=dy$, $z-z'=dz$, the equation of the complex may be written in the differential form used by Lie,

$$xdy-ydx+hdz=0.$$

This equation is satisfied, ρ being constant, by

$$x=r\cos\phi,\ y=r\sin\phi,\ z=-\frac{r^2}{h}\phi,$$

that is, *the lines of the complex are tangent lines to a series of helices, whose axes coincide with the axis of the complex.*

Ex. 1. A *curve of a linear complex* is defined as one whose tangent lines belong to the complex, therefore the coordinates of its points, when expressed as functions of a parameter, satisfy the equation

$$ydx - xdy - hdz = 0.$$

If α, β, γ be the direction-cosines of the tangent line, l, m, n those of the principal normal, and λ, μ, ν those of the bi-normal, then

$$y\alpha - x\beta - h\gamma = 0,$$

and by differentiating with regard to s, and using the first of the Frenet-Serret formulas (Art. 368 (a)),

$$yl - xm - hn = 0,$$

therefore $\lambda : \mu : \nu = y : - x : - h$.

Hence *the polar plane of a point is the osculating plane thereat of all the curves through the point.*

Ex. 2. If we use the equations (Art. 368 (a))

$$\frac{n}{\tau} = \frac{d\nu}{ds} = \alpha\,\frac{d\nu}{dx} + \beta\,\frac{d\nu}{dy} + \gamma\,\frac{d\nu}{dz}$$

we find since $n = \lambda\beta - \mu\alpha$,

$$\frac{1}{\tau} = \frac{h}{x^2 + y^2 + k^2} = \frac{\nu^2}{h}.$$

Hence *all the curves through a given point have the same torsion thereat*, and as Professor MacWeeney has pointed out, *the torsion is proportional to the square of the sine of the angle between the polar plane and the axis.*]

455. Let us pass now to a complex of the second degree, the *quadratic* complex; that is to say, the system of lines whose six coordinates are connected by a relation of the second degree. Then, from what has been said, all the lines of the complex which lie in a given plane envelope a conic, and those which pass through a given point form a cone of the second order. We may consider the assemblage of conics corresponding to a system of parallel planes, and obtain thus what Plücker calls an *equatorial surface* of the complex; or, more generally, the assemblage of conics corresponding to planes which all pass through a given line, obtaining thus Plücker's *complex surface.* It is easy to see that the given line is a double line on the surface, and that the surface is of the fourth degree, its section by one of the planes consisting of the line twice, and of the conic corresponding to the plane. The surface will be of the fourth class, and Plücker shows also that it has eight double points.

[The lines of the complex through a given point lie on a

quadric cone; and the *singular surface* of the complex is the *locus of points at which these cones break up into planes.* The lines of the complex in a given plane envelope a conic and it may be shown that *the envelope of planes for which those conics reduce to pairs of points coincides with the singular surface.* This surface is known as Kummer's quartic and will receive attention in subsequent Articles.]

SINGULAR LINES, POINTS, PLANES, AND SURFACES OF COMPLEXES OF ANY DEGREE.*

[455*a*. If $\phi=0$ be the equation of a complex of the n^{th} degree, the lines through a fixed point A describe a cone of the n^{th} degree, and the lines in a plane envelope a curve of the n^{th} class. If A, B, C be any three neighbouring points their corresponding cones have points in common, and if P be one of them whose corresponding cone is S, then S has PA, PB, PC as generators, and hence this cone must have a double edge which joins P to a point in the immediate vicinity of A, B, C. Of the three lines PA, PB, PC, two in general lie on one sheet and one on the other. A point whose cone has a double edge is called a *singular point* and its double edge a *singular line.*

If $x'y'z'w'$ be such a singular point then replacing p, . . . by $yz'-y'z$, . . . the cone will have a double edge if

$$\frac{d\phi}{dx}=\frac{d\phi}{dy}=\frac{d\phi}{dz}=\frac{d\phi}{dw}=0.$$

Now denoting $\dfrac{d\phi}{dp}$, . . . by $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$, these conditions reduce to

$$y'\phi_3-z'\phi_2+w'\phi_4=0,\quad z'\phi_1-x'\phi_3+w'\phi_5=0,$$
$$x'\phi_2-y'\phi_1+w'\phi_6=0,\quad x'\phi_4+y'\phi_5+z'\phi_6=0, \quad . \qquad (1)$$

or any three of them.

From these we deduce the necessary conditions for a singular line of the complex, viz.:

$$\phi=0,\ \psi=\phi_1\phi_4+\phi_2\phi_5+\phi_3\phi_6=0 \quad . \quad . \qquad (2)$$

It follows therefore from (1) that $\phi_4, \phi_5, \phi_6, \phi_1, \phi_2, \phi_3$ are

* Arts. 455*a*-*o* are due to Mr. R. Russell, F.T.C.D.

the coordinates of a line (*the conjugate line*) which intersects the singular line in the singular point; and the plane of the two lines is called the *singular plane*.

455*b*. The *singular surface* is the locus of a point $x'y'z'w'$ which is the vertex of a cone of the complex having a double edge, and its equation might be obtained as follows:—

If $\psi = 0$ and any two of the equations in (1) be satisfied the remaining two of (1) will be satisfied and also $\phi = 0$. In addition three equations connect x', y', z', w' with p, q, r, s, t, u, hence we have six homogeneous equations from which p, q, r, s, t, u can be eliminated.

By analogous reasoning the singular planes envelope a surface, and it will be shown that this is the singular surface.

455*c*. If $x'y'z'w'$ be the singular point in 455*a* and $x''y''z''w''$ a second point on the double edge PQ of S, then the cones of all points on PQ have the singular plane of PQ as a common tangent plane.

The cone of such a point is

$$\phi\ \{y\ (z'+kz'') - z\ (y'+ky''),\ .\ .\ .\} = 0,$$

and the tangent plane along PQ is

$$x\left(\frac{d\phi}{dx}\right)' + y\left(\frac{d\phi}{dy}\right)' + z\left(\frac{d\phi}{dz}\right)' + w\left(\frac{d\phi}{dw}\right)' = 0$$

$$\text{or } \Sigma\ \{y\ (z'+kz'') - z\ (y'+ky'')\}\ \phi_1 = 0.$$

Making use of the conditions in (1) the tangent plane reduces to $\Sigma\ (yz''-y''z)\ \phi_1 = 0$, and this is the plane of the singular line and its conjugate.

455*d*. Cones of the complex whose vertices lie in a fixed plane contain homogeneously three parameters, and therefore in the usual way envelope a surface.

Suppose for simplicity that the plane is $w'=0$, then one of the cones is

$$\phi\,(yz'-y'z,\ zx'-z'x,\ xy'-x'y,\ -x'w,\ -y'w,\ -z'w) = 0,$$

and the envelope is obtained by eliminating $x'y'z'$ from

$$\frac{d\phi}{dx'}=0 \quad \frac{d\phi}{dy'}=0 \quad \frac{d\phi}{dz'}=0.$$

But if $xyzw$ be regarded as fixed and $x'y'z'$ as variable, these conditions express that the cone S' whose vertex is $xyzw$ meets $w'=0$ in a curve having a double point, and this cannot be true unless S' has a double edge. Hence the cones of the complex whose vertices lie in a fixed plane envelope the singular surface, and from 455*c* we see that the singular plane is the tangent plane to the enveloping cone where it is met by two consecutive cones. Hence the locus of singular points, the envelope of singular planes, and the envelope of cones of the complex are the same singular surface, and the singular plane touches the surface where the singular line to which the plane corresponds meets its conjugate. We might have obtained the same results if we had developed the property that lines of the complex lying in a plane envelope a curve of the n^{th} class, but it is left as an exercise in tangential coordinates.

455*e*. *Degree and Class of the Singular Surface.*—The number of singular points on a given line is obviously the same as the number of singular planes through it, and these cases are both included in the conditions that a singular line and its conjugate meet the given line. If p' . be the given line the conditions are

$$\Sigma ps=0, \quad \phi=0, \quad \Sigma\phi_1\phi_4=0, \quad \Sigma ps'=0, \quad \Sigma p'\phi_1=0.$$

There are therefore $4n(n-1)^2$ cases in all, and the order and class are each $2n(n-1)^2$.

THE QUADRATIC COMPLEX

$$a_{11}p^2+\ .\ .\ .+2a_{12}pq+\ .\ .\ .+2a_{35}rt+\ .\ .\ .=0.$$

455*f*. The cone of the complex with a given vertex is a quadric cone which in the case of a singular point reduces to two planes. The lines of the complex in a given plane envelope a conic which in the case of a singular plane reduces to a pair of points, and the singular surface is of the fourth degree and class.

The four points in which the surface is met by any line are equi-inharmonic with the four tangent planes through it.

This property may be established by several methods of which the following is one: If $z=0$, $w=0$ be the given line, $a, 1, 0, 0$ any point on it, and $0, 0, \nu, 1$ any plane through it, then the cone of the point is obtained by substituting $-z$, za, $u-ay$, $-aw$, $-w$, 0 for p, q, r, s, t, u in $\phi=0$, and if $\nu z+w=0$ is a tangent plane to this cone then

$$a_{33}\left[a_{22}a^2-2a_{12}a+a_{11}-2\nu\left\{-a_{24}a^2+\overline{a_{14}-a_{25}}a+a_{15}\right\}\right.$$
$$\left.+\nu^2(a_{44}a^2+2a_{45}a+a_{55})\right]$$
$$-\{a_{23}a-a_{13}+\nu(a_{34}a+a_{35})\}^2=0,$$

a quadri-quadric function of a, ν.

If the two values of ν are equal the four values of a give singular points on the line, and if the two values of a are equal the four values of ν give singular planes through the line. But the two biquadratics are known to be equi-anharmonic, hence the proposition is proved.

455*g*. *The Singular Quartic has Sixteen Double Points and Sixteen Planes of Contact.* The conic of the complex in any singular plane reduces in general to a distinct pair of points. Are there any cases in which they coincide? If this is so we might expect that all lines in the plane through the point would be singular, and that therefore each of the lines would touch the surface at the singular point on it, and therefore the plane would touch along a conic section.

Suppose $pqrstu$ a singular line, $p_1q_1r_1s_1t_1u_1$ its conjugate, then if $\Sigma\dfrac{d\phi_{11}}{dp_1}\dfrac{d\phi_{11}}{ds_1}=0$ the conjugate line is also singular. Let $p_2q_2r_2s_2t_2u_2$ be its conjugate, and denote by $\overline{12}$ the quantity Σp_1s_2, then we have the following conditions:

$$\Sigma ps=0$$
$$\phi_{00}=0 \text{ involving } \overline{01}=0$$
$$\phi_{01}=0 \text{ involving } \overline{11}=0,\ 02=0$$
$$\phi_{02}= \quad \Sigma p_1s_2=\overline{12}=\phi_{11}=0$$
$$\phi_{12}= \quad \Sigma p_2s_2=\overline{22}=0.$$

Now obviously the lines 0, 1, 2 either lie in a plane or meet in a point, and considering the former case any line in the plane is $\lambda p + \mu p_1 + \nu p_2 \ldots$ If it belongs to the complex, $\nu^2\phi_{22} = 0$, and therefore the two points in the singular plane coincide, or, all lines in the complex and in the plane pass through a point.

Again the line $\lambda p + \mu p_1 \ldots$ is singular, and its conjugate is $\lambda p_1 + \mu p_2, \ldots$, and the former line touches the quartic where it is met by the latter, and hence the plane touches the quartic along a conic.

The five equations of condition give thirty-two solutions, of which sixteen refer to planes of contact, and sixteen to double points.

455*h*. *The Equation of the Singular Quartic.* From the theorem that two quadratic expressions in n variables can be reduced simultaneously to the sums of n squares of linear functions of these variables, it can be immediately inferred that the most general quadratic complex can be put in the form

$$\phi = a_{11}p^2 + a_{22}q^2 + a_{33}r^2 + a_{44}s^2 + a_{55}t^2 + a_{66}u^2 + 2a_{14}ps + 2a_{25}qt + 2a_{36}ru = 0.$$

The singular quartic is the envelope of cones whose vertices are in the plane $w' = 0$. Substituting for $p \ldots$ $yz' - y'z,\ zx' - z'x,\ xy' - x'y,\ -x'w,\ -y'w,\ -z'w$ and forming the envelope, w^2 divides out, and we find for the equation

$$\begin{aligned} &a_{44}a_{22}a_{33}x^4 + a_{55}a_{33}a_{11}y^4 + a_{66}a_{11}a_{22}z^4 + a_{44}a_{55}a_{66}w^4 \\ &\quad + \Sigma\,(a_{11}y^2z^2 + a_{44}x^2w^2)(a_{22}a_{55} + a_{33}a_{66} - a^2) \\ &\quad + 2xyzw\,(\alpha\beta\gamma + a_{11}a_{44}\alpha + a_{22}a_{55}\beta + a_{33}a_{66}\gamma) = 0 \end{aligned}$$

where $\alpha = a_{25} - a_{36},\ \beta = a_{36} - a_{14},\ \gamma = a_{14} - a_{25}$.

Writing $x = \sqrt[4]{a_{11}a_{55}a_{66}}\ X,\ y = \sqrt[4]{a_{22}a_{66}a_{44}}\ Y,\ z = \sqrt[4]{a_{33}a_{44}a_{55}}\ Z,$

$$w = \sqrt[4]{a_{11}a_{22}a_{33}}\ W,$$

$$a_{11}a_{44} = a^2,\ a_{22}a_{55} = b^2,\ a_{33}a_{66} = c^2,$$

and for convenience restoring small letters, it reduces to

$$\begin{aligned} &x^4 + y^4 + z^4 + w^4 + 2\lambda\ (y^2z^2 + x^2w^2) + 2\mu\ (z^2x^2 + y^2w^2) \\ &\quad + 2\nu\ (x^2y^2 + z^2w^2) + 4\kappa xyzw = 0 \qquad \text{where} \qquad (3) \end{aligned}$$

$$\alpha^2 = b^2 + c^2 - 2bc\lambda,\ \beta^2 = c^2 + a^2 - 2ca\mu,\ \gamma^2 = a^2 + b^2 - 2ab\nu,$$

$$2\kappa abc = \alpha\beta\gamma + a^2\alpha + b^2\beta + c^2\gamma,\ \alpha + \beta + \gamma = 0.$$

It is easily seen that a relation exists between $\lambda,\ \mu,\ \nu,\ \kappa$.

The conditions just obtained suggest taking four points $ABCD$ in space such that $BC = a$, $CA = \beta$, $AB = \gamma$, $AD = a$, $BD = b$, $CD = c$ and the cosines of the angles at D as λ, μ, ν.

If the points were real $ABCD$ would obviously lie in an ordinary plane and $1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu = 0$. But this is not the only interpretation. It is easy to verify that if the plane ABC touch the circle at infinity then $AB + BC + CA \equiv 0$ however ABC are chosen, and $\cos A = \cos B = \cos C = -1$. The expression for 6 vol $ABCD$ is $abc\sqrt{1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu}$, and putting $\xi = BAD$, $\eta = CAD$, it is equal to

$$\begin{aligned} a\beta\gamma\sqrt{1 - \cos^2\xi - \cos^2\eta - \cos^2 C + 2\cos\xi\cos\eta\cos C} \\ = \pm\, a\beta\gamma\sqrt{-1}(\cos\xi + \cos\eta) \\ = \pm\frac{\sqrt{-1}}{2}(a\beta\gamma + a^2 a + b^2\beta + c^2\gamma). \end{aligned}$$

Therefore $\kappa^2 + 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu = 0$ . . . . (4)

It may easily be shown that this is the necessary and sufficient condition that the singular quartic (3) shall have double points, that is that the equations

$$\left.\begin{aligned} x(x^2 + \nu y^2 + \mu z^2 + \lambda w^2) + Kyzw &= 0 \\ y(\nu x^2 + y^2 + \lambda z^2 + \mu w^2) + Kzxw &= 0 \\ z(\mu x^2 + \lambda y^2 + z^2 + \nu w^2) + Kxyw &= 0 \\ w(\lambda x^2 + \mu y^2 + \nu z^2 + w^2) + Kxyz &= 0 \end{aligned}\right\} \quad . \quad . \quad (5)$$

are simultaneously satisfied.

And if the quartic has one double point it will have sixteen, obtained by permuting and changing the signs of x, y, z, w so as to leave $y^2z^2 + x^2w^2$, $z^2x^2 + y^2w^2$, $x^2y^2 + z^2w^2$ $xyzw$ unaltered.

455*i*. *Planes of Contact and Double Points of the Singular Quartic.* The equation of the quartic being unaltered by interchanging x, y and also z, w, etc., if $ax + \beta y + \gamma z + \delta w = 0$ be one plane of contact, so also are

$\delta x + \gamma y + \beta z + aw = 0$, $\gamma x + \delta y + az + \beta w = 0$,

$\beta x + ay + \delta z + \gamma w = 0$.

Denoting these by P, L, M, N respectively, and

$\dfrac{x^2+y^2+z^2+w^2}{2}$, $yz+xw$, $zx+yw$, $xy+zw$ by S, U, V, W,

and $\dfrac{a^2+\beta^2+\gamma^2+\delta^2}{2}$, $\beta\gamma+a\delta$, $\gamma a+\beta\delta$, $a\beta+\gamma\delta$ by σ, θ_1, θ_2, θ_3,

$$LMNP=(abcdfghlmn)\ (U,\ V,\ W,\ S)^2+kxyzw, \qquad (6)$$

where obviously $d=4a\beta\gamma\delta$, $l=\theta_2\theta_3$.

Now the double transformation

$$x=\rho\ (y'+ix'),\ y=\rho\ (x'+iy'),\ z=\rho\ (w'+iz'),\ w=\rho\ (z'+iw')$$
$$a'=\rho\ (\beta+ia),\ \beta'=\rho\ (a+i\beta),\ \gamma'=\rho\ (\delta+i\gamma),\ \delta'=\rho\ (\gamma+i\delta)$$

involves $LMNP=L'M'N'P'$,

$$U=2i\rho^2V',\ V=2i\rho^2U',\ W=2i\rho^2S',\ S=2i\rho^2W',$$

and therefore the equation (6) becomes

$$L'M'N'P'=-4\rho^4\ (a,\ b,\ c,\ d,\ f,\ g,\ h,\ l,\ m,\ n)^2\ \ (V',\ U',\ S',\ W')^2 \\ -k\rho^4\ (V'^2+U'^2-4x'y'z'w').$$

Comparing we have $-4\rho^4c=4a'\beta'\gamma'\delta'=-4\rho^4\,(a^2+\beta^2)\ (\gamma^2+\delta^2)$,

$$-8\rho^4f=2l'=2\theta_2'\theta_3'=-8\rho^4\theta_1\sigma,$$
$$\text{or } c=(a^2+\beta^2)\ (\gamma^2+\delta^2),\ f=\theta_1\sigma,\ l=\theta_2\theta_3.$$

Substituting these and analogous values if desired in (6) we see that the six products in pairs of U, V, W, S are the terms of the square

$$\left(\frac{S}{\sigma}+\frac{U}{\theta_1}+\frac{V}{\theta_2}+\frac{W}{\theta_3}\right)^2\equiv\Psi^2,$$

and we have the identity

$$\Psi^2-LMNP\equiv R\,[x^4+y^4+z^4+w^4+2\lambda\ (y^2z^2+x^2w^2) \\ +2\mu\ (z^2x^2+y^2w^2)+2\nu\ (x^2y^2+z^2w^2)+4Kxyzw] \qquad (7)$$
$$\text{or } \Psi^2-LMNP\equiv R\ \Phi,$$

where R, λ, μ, ν, K are to be found, if desired, in terms of a, β, γ, δ.

Now $\Psi^2-LMNP=0$ has double points where $L=0$, $M=0$ meets $\Psi=0$, that is at the points $(a,\ \beta,\ -\gamma,\ -\delta)$ and $a,-\beta,\ \gamma,-\delta$. It follows therefore that $\Phi=0$ has these as double points, and

$$\frac{d\Phi}{dx}=\frac{d\Phi}{dy}=\frac{d\Phi}{dz}=\frac{d\Phi}{dw}=0$$

are satisfied by a, β, $-\gamma$, $-\delta$.

These give equations identical with (5) in 455*h* from which the value of λ, μ, ν, K can easily be found. In fact

$$\Phi \equiv \begin{vmatrix} \Sigma x^4, & 2(y^2z^2+x^2w^2), & 2(z^2x^2+y^2w^2), & 2(x^2y^2+z^2w^2), & 4xyzw \\ \alpha^3 & \alpha\delta^2 & \alpha\gamma^2 & \alpha\beta^2 & \beta\gamma\delta \\ \beta^3 & \beta\gamma^2 & \beta\delta^2 & \beta\alpha^2 & \gamma\alpha\delta \\ \gamma^3 & \gamma\beta^2 & \gamma\alpha^2 & \gamma\delta^2 & \alpha\beta\delta \\ \delta^3 & \delta\alpha^2 & \delta\beta^2 & \delta\gamma^2 & \alpha\beta\gamma \end{vmatrix}.$$

Finally $\dfrac{L^2+M^2+N^2+P^2}{2}$, $MN+LP$, $NL+MP$, $LM+NP$ are linear functions of S, U, V, W, hence $\Psi^2-LMNP=0$ reduces to the form

$$\{L^2+M^2+N^2+P^2+2f(MN+LP)$$
$$+2g(NL+MP)+2h(LM+NP)\}^2-16kLMNP=0,$$

and a relation exists between the coefficients f, g, h, k for double points. It is $k+1-f^2-g^2-h^2+2fgh=0$.

455*j*. *The Singular Quartic is a General Kummer's Quartic —that is one having Sixteen Double Points.*

If a quartic have sixteen nodes the tangent cone from one of them is of the sixth degree, and it has as double edges the lines joining that node to the fifteen others. It therefore breaks up into six planes, and each of these planes touches the surface along a conic.

Each plane through the node is met by the five others in a nodal line, that is a line joining two double points, so that six double points lie in every plane of contact, two planes of contact have two double points on their line of intersection, and through the line joining two double points two planes of contact can be drawn.

By drawing four planes of contact that form a tetrahedron $ABCD$ there are two double points on each of the lines DA, DB, DC, BC, CA, AB. If we choose those on DA, DB, DC and one point on each of the conics in which the quartic is touched by the planes DBC, DCA, DAB, we can describe a quadric through these nine points, and it will therefore contain the conics in the planes DBC, DCA, DAB. It contains six points on the plane of contact of ABC and therefore also the conic, and therefore the quartic is reducible to the form

$$\Psi^2=16kxyzw.$$

Ψ may be taken of the form

$$x^2+y^2+z^2+w^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw,$$

and the double points on the six edges are

$$\begin{array}{lll} x=0 & w=0 & (y-az)\,(y-a'z)=0 \\ y=0 & w=0 & (z-\beta x)\,(z-\beta' x)=0 \\ z=0 & w=0 & (x-\gamma y)\,(x-\gamma' y)=0 \\ y=0 & z=0 & (x-\lambda w)\,(x-\lambda' w)=0 \\ z=0 & x=0 & (y-\mu w)\,(y-\mu' w)=0 \\ x=0 & y=0 & (z-\nu w)\,(z-\nu' w)=0. \end{array}$$

Now through the point $0\beta 10$ in addition to the planes of contact $x=0$, $w=0$ four others can be drawn each of which meets the edges AB, AC, AD in double points on them, therefore the double points connected with the quantities $a\ \beta\ \gamma\ \lambda$ are coplanar: hence $a\beta\gamma+1=0$.

Similarly $a\beta'\gamma'+1=0, \quad \beta\gamma'a'+1=0, \quad \gamma a'\beta'+1=0,$

therefore $a=a'$, $\beta=\beta'$, $\gamma=\gamma'$, or

$$a=-a',\ \beta=-\beta',\ \gamma=-\gamma'$$

which involve $f=l$, $g=m$, $h=n$, or

$$f=-l,\ g=-m,\ h=-n,$$

and Ψ may be taken

$$x^2+y^2+z^2+w^2+2f\,(yz+xw)+2g\,(zx+yw)+2h\,(xy+zw).$$

455*k*. *Condition that a line PQRSTU shall be a Conjugate Line of* $\phi=0$.

Evidently $\dfrac{\phi_1}{S}=\dfrac{\phi_2}{T}=\dfrac{\phi_3}{U}=\dfrac{\phi_4}{P}=\dfrac{\phi_5}{Q}=\dfrac{\phi_6}{R}=k$ . . (1)

and $pS+qT+rU+sP+tQ+uR=0.$

Eliminating $pqrstu$ we obtain as the condition a symmetrical determinant equation

$$\Phi\equiv\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & S \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ a_{61}, & a_{62}, & a_{63}, & a_{64}, & a_{65}, & a_{66}, & R \\ S, & T, & U, & P, & Q, & R, & \cdot \end{vmatrix}=0\ . \qquad (2)$$

This might be called the *conjugate complex* of $\phi=0$. It may be written

$$A_{11}S^2+\ .\ .\ .\ +A_{44}P^2+\ .\ .\ .\ +2A_{12}ST+\ .\ .\ .\ +2A_{15}SQ$$
$$+\ .\ .\ .\ =0.$$

where A_{mn} is the minor of a_{mn} in the discriminant of ϕ.

Cosingular Complexes.—From the equations (1), (2) it appears on solving for p, q, r, s, t, u, or from general considerations, that

$$\Delta p = k(A_{11}S + A_{12}T + A_{13}U + A_{14}P + A_{15}Q + A_{16}R)$$

$$= \tfrac{1}{2}k\frac{d\Phi}{dS} = k\Phi_4, \text{ etc.}$$

hence every singular line of $\phi = 0$ is a conjugate line of $\Phi = 0$, and every singular line of $\Phi = 0$ is a conjugate line of $\phi = 0$, and the two complexes $\phi = 0$, $\Phi = 0$ have the same singular surfaces.

The original complex might have been written

$$\phi + 2\mu(ps + qt + ru) - 0,$$

when the conjugate complex $\Phi_\mu = 0$ would have been the same determinant equation as above, only that a_{14}, a_{25}, a_{36} would have been replaced by $a_{14} + \mu$, $a_{25} + \mu$, $a_{36} + \mu$. We thus see that all complexes $\Phi_\mu = 0$ have the same singular surface as $\phi = 0$.

455*l*. *Double Tangent Lines of the Singular Surface.* If a conjugate line be given there is usually one singular line to which it corresponds. If $p'\ q'\ r'\ s'\ t'\ u'$ be a singular line we have to consider the equations

$$\phi_n = \phi_n' \ (n = 1, 2, 3, 4, 5, 6), \ ps + qt + ru = 0,$$

and in general the only solution is

$$p, q, r, s, t, u = p', q', r', s', t', u'.$$

But if $\phi_1\ \phi_2\ \phi_3\ \phi_4\ \phi_5\ \phi_6$ are identically connected by a linear relation, that is if the discriminant of ϕ vanishes, the seven equations reduce to five of the first and one of the second degree, of which there are two distinct solutions, and the conjugate line passes through the point of contact of each tangent plane.

Now k may be any one of six values if the discriminant of $\phi - 2k(ps + qt + ru) = 0$ vanishes, and hence for any of these six forms of the quadratic complex the conjugate lines are double tangents to the singular surface.

455*m*. *The Principal Linear Complexes.* Suppose $pqrstu$ are any six quantities, not coordinates of a line, satisfying the equations

$$\frac{\phi_4}{p} = \frac{\phi_5}{q} = \frac{\phi_6}{r} = \frac{\phi_1}{s} = \frac{\phi_2}{t} = \frac{\phi_3}{u} = k,$$

then k is a root of the discriminant of $\phi - 2k(ps + qt + ru) = 0$.

There are therefore six sets of quantities $p_1, \ldots, p_2 \ldots, p_6 \ldots$ corresponding to the roots $k_1, \ldots k_6$, and they satisfy the following relations:—

$\phi_{mn} = k_m\overline{mn} = k_n\overline{mn}$, therefore $\phi_{mn} = 0, \overline{mn} = 0$, where m, n have any values 1, 2, 3, 4, 5, 6, but $m \neq n$.

Also $\phi_{mm} = k_m\overline{mm} = 2k_m(p_m s_m + q_m t_m + r_m u_m)$.

Now consider the complex $ps_1 + qt_1 + ru_1 + sp_1 + tq_1 + ru_1 = 0$.

If $pqrstu$ be any singular line, and $\phi_4, \phi_5, \phi_6, \phi_1, \phi_2, \phi_3$ its conjugate, then the line $\phi_4 - \theta p, \ldots$ which lies in the first of the above linear complexes is such that

$$\phi_{01} - \theta\overline{01} = 0 \text{ or } (k_1 - \theta)\ 01 = 0, \text{ therefore } \theta = k_1.$$

But this line is the conjugate of $pqrstu$ with respect to $\phi - 2k_1\ (ps + qt + ru) = 0$; and this is a case in which the line $\phi_4 - k_1 p, \ldots$ is conjugate to two distinct singular lines. We see, therefore, that if from a singular point lines be drawn in the singular plane tangents elsewhere to the singular surface they belong to the six principal linear complexes

$$\overline{om} = 0\ (m = 1, 2, 3, 4, 5, 6).$$

If we transform the equations $\phi = 0$, $\psi = 0$, $ps + qt + ru = 0$ by the substitutions

$$p = \frac{p_1}{\sqrt{11}}P + \frac{p_2}{\sqrt{22}}Q + \ldots + \frac{p_6}{\sqrt{66}}U,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$s = \frac{s_1}{\sqrt{11}}P \quad + \ldots + \quad \frac{s_6}{\sqrt{66}}U,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

or their equivalents $P\sqrt{11} = ps_1 + qt_1 + ru_1 + sp_1 + tq_1 + ru_1$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

we obtain $\phi \equiv k_1P^2 + k_2Q^2 + k_3R^2 + k_4S^2 + k_5T^2 + k_6U^2 = 0,$

$$\psi \equiv k_1^2P^2 + \ldots + k_6^2U^2 = 0,$$

$$2(ps+qt+ru)=P^2+Q^2+R^2+S^2+T^2+U^2=0,$$

in which $P=0$, $Q=0$ . . . are the six principal linear complexes.

From these equations we may write

$$P=\frac{\sqrt{u-k_1}\sqrt{v-k_1}}{\sqrt{f'(k_1)}} \ . \ . \ . \ ,$$

and finally obtain parametric expressions for the singular lines, viz.

$$p=\sum_1^6 \frac{p_m\sqrt{u-k_m}\ \sqrt{v-k_m}}{\sqrt{\overline{mm}\ f'(k_m)}},\quad q=\sum_1^6 \frac{q_m\sqrt{u-k_m}\ \sqrt{v-k_m}}{\sqrt{\overline{mm}\ f'(k_m)}}.$$

With the introduction of suitable multipliers attached to the groups $(p_1\ q_1\ r_1\ s_1\ t_1\ u_1)$, $(p_2\ q_2\ r_2\ s_2\ t_2\ u_2)$ . . . , which do not require explanation, we may more simply take the representation of singular lines and their conjugates in the forms

$$\left(\sum_1^6 p_m\sqrt{u-k_m}\sqrt{v-k_m}, \ . \ .\right), \left(\sum_1^6 p_mk_m\sqrt{u-k_m}\sqrt{v-k_m}, \ . \ . \ .\right),$$

in which it follows from what precedes that

$$\overline{mn}=0 \quad \sum_1^6 k_m^\rho . \overline{mm}=0 \quad \rho=0,\ 1,\ 2,\ 3,\ 4.$$

The case of $u=v$ is special, for here the singular lines are

$$p_1\ (v-k_1)\ \pm\ p_2\ (v-k_2)\ \ . \ . \ . \pm\ p_6\ (v-k_6). \ . \ . \ ,$$

and their conjugates are

$$p_1k_1\ (v-k_1)\ \pm\ p_2k_2\ (v-k_2)\ \ . \ . \ . \pm\ p_6k_6\ (v-k_6). \ . \ . \ .$$

Since there are 32 ways of distributing the signs it follows that these singular lines lie in one or other of 32 pencils, and that all the singular lines in each pencil are intersected by all the conjugate lines, and finally that the singular lines and their conjugates are either coplanar or concurrent. The former correspond to the planes of contact of the singular surface, and the latter to the nodes. To actually divide the 32 special cases into planes of contact and nodes the following proposition may be used: If 1, 2, 3 be three coplanar, and 4, 5, 6 three concurrent lines, the point (4 5 6) lies or does not lie in the plane (1 2 3) according as

$$\left|\begin{matrix} 14, 15, 16 \\ 24, 25, 26 \\ 34, 35, 36 \end{matrix}\right| \text{ does or does not vanish.}$$

Now suppose that $\sum\limits_1^6 p_m, \sum\limits_1^6 q_m \ldots$ and its conjugate determine a coplanar group, and that $\sum\limits_1^6 p_m i_m, \sum\limits_1^6 q_m i_m \ldots$ where $i_m = \pm 1$ forms a concurrent group in which the point of concurrence is outside the plane, then applying the above condition we find that the determinant becomes

$$\left|\begin{matrix} \Sigma i_m . \overline{mm} & \Sigma k_m i_m . \overline{mm} & \Sigma k_m^2 i_m . \overline{mm} \\ \Sigma k_m i_m . \overline{mm} & \Sigma k_m i_m . \overline{mm} & \Sigma k_m^3 i_m . \overline{mm} \\ \Sigma k_m i_m . \overline{mm} & \Sigma k_m^3 i_m . \overline{mm} & \Sigma k_m^4 i_m . \overline{mm} \end{matrix}\right|$$

Since $\Sigma k^\rho \overline{mm} = 0$ $(\rho = 0, 1, 2, 3, 4)$ this reduces at once to a vanishing determinant unless three of the i's are positive and three negative. There are ten such arrangements giving rise to ten nodes outside the plane of contact: the remaining six nodes lie on the conic of contact.

Similarly starting with any node we can at once write down the ten planes of contact that do not contain it, and there is no difficulty in arranging the 32 cases into their groups of nodes and planes of contact.

455n. When a congruence is defined as the system of rays common to two complexes whose equations are given in line coordinates the existence of the focal points and planes (Art. 457) may be deduced as follows. The two complexes being $f = 0$, $\phi = 0$, we have also $ps + qt + ru = 0$, and if two consecutive lines intersect, δp, δq, δr, δs, δt, δu are connected by the following equations:—

$$\Sigma f_1 \delta p = 0,\ \Sigma \phi_1 \delta p = 0,\ \Sigma s \delta p = 0,\ \Sigma \delta p \delta s = 0.$$

We may therefore regard δp, δq, δr, δs, δt, δu as defining a line P which intersects L (p, q, r, s, t, u), and the two lines A, B whose coordinates are $kf_4 + \phi_4$, $kf_5 + \phi_5$, $\ldots$ $kf_3 + \phi_3$ where

$$\Sigma\ (kf_1 + \phi_1)\ (kf_4 + \phi_4) = 0.$$

Now L intersects each of the lines A, B, therefore P is

concurrent and coplanar with either L and A or L and B. These are the points (LA, LB) in which a line is met by a consecutive line (the focal points), and the planes (LA, LB) are those of the consecutive intersecting lines (the focal planes).

455*o*. *Congruence of the Quadratic and Linear Complexes*, viz.

$$\phi = 0,\ \Sigma S_1 p = 0 \text{ where } \Sigma P_1 S_1 \neq 0.$$

Noting that $f_4, f_5, f_6, f_1, f_2, f_3$ of the last article are here $P_1\ Q_1\ R_1\ S_1\ T_1\ U_1$, we shall define a new line by the equations $\phi_1 + kS_1 = \mu S\ \ldots\ \phi_4 + kP_1 = \mu P\ \ldots$ and therefore $\Sigma Sp = 0$. Solving for $pqrstu$ we have

$$\Delta p = \tfrac{1}{2}\left(\mu \frac{d\Phi_{00}}{dS} - k\frac{d\Phi_{11}}{dS_1}\right)\ \ldots\ \Delta s = \left(\mu \frac{d\Phi_{00}}{dP} - k\frac{d\Phi_{11}}{dP_1}\right)$$

where $\Phi = 0$ is the conjugate complex of $\phi = 0$.
Since $\Sigma S_1 p = 0$, $\Sigma Sp = 0$ we have

$$\mu\Phi_{00} - k\Phi_{01} = 0 \qquad \mu\Phi_{01} - k\Phi_{11} = 0$$

therefore $PQRSTU$ satisfy the complex

$$\Theta \equiv \Phi_{00}\Phi_{11} - \Phi_{01}{}^2 = 0,$$

which possesses the property that *its discriminant vanishes*.

$$\text{Again } \Theta_4 \equiv \tfrac{1}{2}\left(\frac{d\Phi_{00}}{dS}\Phi_{11} - \Phi_{01}\frac{d\Phi_{11}}{dS_1}\right) = \tfrac{1}{2}\left(\frac{d\phi_{00}}{\mu dS} - k\frac{d\Phi_{11}}{dS_1}\right) = \Delta p,$$

therefore $\Theta_1\Theta_4 + \Theta_2\Theta_5 + \Theta_3\Theta_6 = 0$, or P, Q, R, S, T, U is a singular line of $\Theta = 0$, and $pqrstu$ is its conjugate, and is therefore a bitangent to the singular surface of $\Theta = 0$.

If we compare we see that $PQRSTU$ and its conjugate with respect to $\Theta = 0$ form precisely in this the same pencil of lines as $pqrstu$ and $\delta p\delta q\delta r\delta s\delta t\delta u$ in the last article, and we recognize therefore *the identity of the focal surface of the congruence with the singular surface of*

$$\Phi_{00}\Phi_{11} - \Phi_{01}{}^2 = 0.]$$

SECTION II.—RECTILINEAR CONGRUENCES.

455*p*. We have a rectilinear congruence of the second order when we have two equations each of the first degree between the six coordinates ; or, in other words, the congruence con-

sists of the lines common to two given complexes. We may evidently for either of the two given equations

$$Ap + Bq + \&c. = 0,\ A'p + \&c. = 0,$$

substitute any equation of the form

$$(A + kA')\,p + \&c. = 0;$$

and then determine k, so that this equation shall express that every line of the congruence meets a given line. We have thus a quadratic equation for k, and it appears that the congruence consists of the system of lines which meet two fixed directing lines. Any four lines then determine a congruence of this kind; for (see Art. 57*d*) we have two transversals which meet all four lines,* and the congruence consists of all the lines which meet the two transversals. An exception occurs when these two transversals unite in a single one; or, what is the same thing, when the quadratic equation just mentioned has two equal roots. The lines of the congruence, then, all meet the single transversal; but, of course, another condition is required; and by considering the transversal as the limit of two distinct lines we arrive at the condition in question; in fact the congruence consists of lines each meet-

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* The hyperboloid determined by any three of the lines (see Art. 113) meets the fourth in two points through which the transversals pass. If the hyperboloid touches the fourth line, the two transversals reduce to a single one and it is evident that the hyperboloid determined by any three others of the four lines also touches the remaining one. This remark, I believe, is Cayley's. If we denote the condition that two lines should intersect by (12), then the above condition that four lines should be met by only one transversal is expressed by equating to nothing the determinant

$$\begin{vmatrix} - & (12), & (13), & (14) \\ (21), & - & (23), & (24) \\ (31), & (32), & - & (34) \\ (41), & (42), & (43), & - \end{vmatrix}.$$

The vanishing of the determinant formed in the same manner from five lines is the condition that they may all meet a common transversal. The vanishing of the similar determinant for six lines expresses that they all belong to a linear complex, which has been called the "involution of six lines"; and occurs when the lines can be the directions of six forces in equilibrium. The reader will find several interesting communications on this subject by Sylvester and Cayley, and by Chasles, in the *Comptes Rendus* for 1861, *Premier Semestre.*

ing a given line, and such that considering the common point of the given line and a line of the congruence, and the common plane of the same two lines, the range of points corresponds homographically with the pencil of planes.

[455q. Instead of treating a rectilinear congruence as a system of rays whose coordinates satisfy two relations, we might define it by the property that in general a finite number of rays, or one only, passes through any point in space; and this definition is more convenient in dealing with certain *differential* properties.

There are now two methods of procedure. In the first method, used by Hamilton, we regard the direction-cosines (l, m, n) of the ray through any point (P, x, y, z) as determinate functions of x, y, z, connected by the relation $l^2+m^2+n^2=1$. In general l, m, n, if taken as the direction-cosines of the tangent line to a curve through P will define a *congruence of curves*, a finite number of curves passing through each point of space. But in order that the congruence may be *rectilinear* the values of l, m, n must not vary as we proceed along the direction they determine. Now if u be any function of x, y, z we have

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy + \frac{du}{dz}\,dz,$$

and therefore if u does not vary along the direction l, m, n,

$$l\frac{du}{dx} + m\frac{du}{dy} + n\frac{du}{dz} = 0.$$

If then we substitute l, m, n for u in this equation we get three conditions to be satisfied in order that the congruence may be rectilinear. But we have also $l^2 + m^2 + n^2 = 1$, and if both sides of this equation be differentiated with regard to x, y, z in turn, it is easy to see that the conditions are equivalent to

$$\frac{l}{\dfrac{dn}{dy}-\dfrac{dm}{dz}} = \frac{m}{\dfrac{dl}{dz}-\dfrac{dn}{dx}} = \frac{n}{\dfrac{dm}{dx}-\dfrac{dl}{dy}}$$

with $l^2 + m^2 + n^2 = 1$.

Ex. 1. From the preceding condition it follows at once that the necessary and sufficient conditions that l, m, n should represent a congruence of normals to a family of surfaces are

$$\frac{dn}{dy} - \frac{dm}{dz} = \frac{dl}{dz} - \frac{dn}{dx} = \frac{dm}{dx} - \frac{dl}{dy} = 0,$$

or in other words that

$$ldx + mdy + ndz$$

should be a perfect differential. See Art. 457d.

Ex. 2. The condition that three functions f, g, h, of x, y, z may be proportional at each point to the direction-cosines of a rectilinear congruence is

$$\frac{f\frac{df}{dx} + g\frac{df}{dy} + h\frac{df}{dz}}{f} = \frac{f\frac{dg}{dx} + g\frac{dg}{dy} + h\frac{dg}{dz}}{g} = \frac{f\frac{dh}{dx} + g\frac{dh}{dy} + h\frac{dh}{dz}}{h}.$$

Ex. 3. Prove that for any rectilinear congruence

$$\left(l\frac{d}{dx} + m\frac{d}{dy} + n\frac{d}{dz}\right)I + IJ = 0$$

where $I = \Sigma\, l\left(\frac{dn}{dy} - \frac{dm}{dz}\right)$, $J = \Sigma\frac{dl}{dx}$.

The second method, used by Kummer and succeeding writers, is an application of Gauss' parametric treatment of surfaces (Art. 377). In fact the rays of a congruence, like the points on a plane or surface, form a "doubly-infinite" manifold. We choose arbitrarily *a director surface* or *surface of reference*, and through each point (x, y, z) thereof draw the corresponding ray (l, m, n). Then x, y, z, l, m, n are functions of two parameters p and q, since x, y, z lies on a known surface and l, m, n depend only on the position of the point xyz.]

[456. *Limit Points, Principal Planes.* There are certain points and planes associated with each ray of a congruence, depending on the limiting relations of the ray to those in its neighbourhood.

Let the equations of a ray through x, y, z be

$$\frac{\xi - x}{l} = \frac{\eta - y}{m} = \frac{\zeta - z}{n}.$$

Then x, y, z, l, m, n, as just explained, may be regarded as functions of two parameters, the point x, y, z moving on the surface of reference. If we take a second ray ($l'm'n'$) through another point ($x'y'z'$) on the surface of reference and consider the line joining a point $x + lr$, $y + mr$, $z + nr$, to a point

$x'+l'r'$, $y'+m'r'$, $z'+n'r'$ on the second ray, then the conditions that the joining line may be perpendicular to both give

$$l\,(x'-x)+m\,(y'-y)+n\,(z'-z)-r+r'\cos\theta=0$$
$$l'(x'-x)+m'\,(y'-y)+n'\,(z'-z)+r'-r\cos\theta=0,$$

where θ is the angle between the rays. If we take the rays indefinitely near, and so replace $x'-x$, $y'-y$, $z'-z$, $r'-r$, by dx, dy, dz, dr, we derive from these the equation

$$-r=\frac{dxdl+dydm+dzdn}{dl^2+dm^2+dn^2} \quad . \quad . \quad (1)$$

which determines the point where one ray is met by the shortest distance from a "consecutive ray" through $x+dx$, $y+dy$, $z+dz$. Replacing dx, dl, etc., by

$$\frac{dx}{dp}dp+\frac{dx}{dq}dq,\quad \frac{dl}{dp}dp+\frac{dl}{dq}dq,\quad \text{etc.}$$

we find

$$-r=\frac{adp^2+(b+b')\,dpdq+cdq^2}{edp^2+2fdpdq+gdq^2}$$

where, using the suffixes 1 and 2 to denote differentiation with regard to p and q,

$$a=l_1x_1+m_1y_1+n_1z_1,\quad b=l_1x_2+m_1y_2+n_1z_2$$
$$b'=l_2x_1+m_2y_1+n_2z_1,\quad c=l_2x_2+m_2y_2+n_2z_2$$
$$e=l_1^2+m_1^2+n_1^2,\quad f=l_1l_2+m_1m_2+n_1n_2,\quad g=l_2^2+m_2^2+n_2^2.$$

Writing t for the ratio $dp:dq$ we have

$$r=-\frac{at^2+(b+b')\,t+c}{et^2+2ft+g} \quad . \quad . \quad . \quad (2)$$

Since the denominator of this fraction is the sum of three squares it cannot change sign, and r therefore cannot become infinite, but will lie between two extreme values; that is to say, *the points on any ray of a congruence where it is met by the shortest distance from a consecutive ray, range on a certain determinate portion of the line*, the extreme points being called by Hamilton the *virtual foci*,* but now more commonly the *limit points* of the ray.

It is easily proved that the values of r for the limit-points are roots of the equation

* First Supplement Trans. R.I.A., XVI, Part I, p. 52.

$$(eg-f^2)\,r^2+\{ag+ec-f\,(b+b')\}r+ac-\tfrac{1}{4}(b+b')^2=0 \quad . \quad . \quad . \quad . \quad (3)$$

and the values of t corresponding to these are roots of

$$\begin{vmatrix} et+f & at+\frac{1}{2}(b+b') \\ ft+g & \frac{1}{2}(b+b')t+c \end{vmatrix}=0.$$

The direction-cosines of the right line perpendicular to a ray lmn, and to its shortest distance from a consecutive ray are proportional to dl, dm, dn, that is to l_1dp+l_2dq, m_1dp+m_2dq, n_1dp+n_2dq, and for the values of $dp : dq$ corresponding to the limit points they are therefore proportional respectively to

$$l_1t+l_2,\ m_1t+m_2,\ n_1t+n_2$$
$$l_1t'+l_2,\ m_1t'+m_2,\ n_1t'+n_2$$

where t and t' are the roots of the preceding quadratic in t. The condition that these two lines should be at right angles is thus

$$ett'+f\,(t+t')+g=0,$$

and the quadratic equation shows that this condition is satisfied. It follows also that *the shortest distances corresponding to the limit points are at right angles.*

The planes containing a ray and these extreme shortest distances are called *the principal planes* of the ray, and we infer that *the principal planes are at right angles.*

Let us now suppose that the consecutive rays corresponding to the limit points are those for which $dp=0$ and $dq=0$, i.e. they correspond to the parametric lines on the surface of reference. Then the roots of the quadratic for t are 0 and ∞, and if we exclude the case (considered in Art. 457*e*) for which a, $b+b'$, c are proportional to e, $2f$, g, we find

$$b+b'=0,\ f=0.$$

Now the direction-cosines of the right line perpendicular to the plane containing a ray and its shortest distance from a consecutive ray, are

$$\frac{l_1t+l_2}{\sqrt{et^2+g}},\quad \frac{m_1t+m_2}{\sqrt{et^2+g}},\quad \frac{n_1t+n_2}{\sqrt{et^2+g}}$$

where $t=\dfrac{dp}{dq}$, and those of the normals to the principal planes are

$$\frac{l_1}{\sqrt{e}}, \frac{m_1}{\sqrt{e}}, \frac{n_1}{\sqrt{e}}, \text{ and } \frac{l_2}{\sqrt{e}}, \frac{m_2}{\sqrt{e}}, \frac{n_2}{\sqrt{e}}.$$

Hence if θ be the angle between the shortest distance and a chosen principal plane (or one of the extreme shortest distance)

$$\cos\theta = \frac{t\sqrt{e}}{\sqrt{et^2+g}}, \qquad \sin\theta = \frac{\sqrt{g}}{\sqrt{et^2+g}}.$$

For the special parameters used the equation (3) determining the distances (r_1, r_2) of the limit points is at once factorized, and we find

$$r_1 = -\frac{a}{e}, \qquad r_2 = -\frac{c}{g}.$$

Now $\quad -r = \dfrac{at^2+c}{et^2+g}$, hence we reach *Hamilton's equation*

$$r = r_1 \cos^2\theta + r_2 \sin^2\theta.]$$

[457. *The focal points, focal planes, focal surface, developables.* When a congruence consists of normals to a surface there are two points (centres of curvature) where a given ray intersects a consecutive ray (Arts. 301, 303, 378). In like manner we shall now prove that, *on each ray of any rectilinear congruence there are two points where it intersects a consecutive ray*, and these are called the *focal points* or *foci* of the ray. This is plainly equivalent to the theorem stated in Art. 453, that *a congruence is a system of bitangents to a certain surface.*

As before let xyz be the coordinates of a point P on any chosen surface of reference and lmn the ray through P. If ρ be the distance of a focal point from P, its coordinates are $x+l\rho$, $y+m\rho$, $z+n\rho$; and since it lies on the consecutive ray $l+dl$, $m+dm$, $n+dn$ through the consecutive point $x+dx$, $y+dy$, $z+dz$, we have

$$dx + l d\rho + \rho dl = 0 \text{ with two similar equations.}$$

Since $dx = x_1 dp + x_2 dq$, $dl = l_1 dp + l_2 dq$, etc., these equations become

$$(x_1+\rho l_1)dp+(x_2+\rho l_2)dq+ld\rho=0$$
$$(y_1+\rho m_1)dp+(y_2+\rho m_2)dq+md\rho=0$$
$$(z_1+\rho n_1)dp+(z_2+\rho n_2)dq+nd\rho=0.$$

Now $ll_1+mm_1+nn_1=0$, $ll_2+mm_2+nn_2=0$; therefore if both sides of these three equations be multiplied by l_1, m_1, n_1, and then by l_2, m_2, n_2, we have by addition

$$(a+e\rho)\ dp+(b+f\rho)\ dq=0$$
$$(b'+f\rho)\ dp+(c+g\rho)\ dq=0$$

where a, b, b', c, e, f, g have the meanings assigned to them in the last article. The elimination of ρ yields an equation for determining the values of $dp : dq$ corresponding to the *two focal points*, namely

$$(eb'-af)\,dp^2+\{ec+f(b'-b)-ag\}\,dpdq+(fc-bg)dq^2=0.$$

The focal distances are formed by eliminating $dp : dq$, and are roots (ρ_1, ρ_2) of the equation

$$\begin{vmatrix} a+e\rho & b+f\rho \\ b'+f\rho & c+g\rho \end{vmatrix}=0$$

$$\text{or } (eg-f^2)\,\rho^2+\{ag+ec-f(b+b')\}\,\rho+ac-bb'=0.$$

Comparing this with the equation (3) in Art. 456 for the distances of the limit points, we have

$$r_1+r_2=\rho_1+\rho_2$$

and therefore, *the point half-way between the limit points coincides with that half-way between the focal points.* This point is called the *middle point* of the ray.

The *focal surface* is the locus of the focal points, and is, like the surface of centres, two sheeted (Art. 306). By the arguments used in Art. 306, it follows that each ray of the congruence touches both sheets at the focal points, and the two *focal planes* being defined as the planes containing a ray and a consecutive ray are tangent planes to the focal surface, which is equally well defined as the envelope of the focal planes.

Since each ray meets two consecutive rays, there are two singly infinite families of *developables* consisting of rays of the congruence. The corresponding cuspidal edges lie on the two

sheets of the focal surface, and each sheet is enveloped by the developables of one family.

Ex. 1. If θ_1, θ_2 be the angles made by the focal planes with the principal plane, prove from Hamilton's equation (last Art.) that

$$\cos 2\theta_1 + \cos 2\theta_2 = 0$$

and hence show that *the focal planes and the principal planes have common bisecting planes.*

Ex. 2. The direction-cosines of the normal to a focal plane are proportional to $(mn_1 - nm_1)s + mn_2 - m_2n$, etc., where s is a root of the quadratic determining the values of $dp : dq$ corresponding to the focal planes.

Ex. 3. A congruence clearly reciprocates into a congruence. The focal planes and points are interchanged and the focal surface reciprocates into the new focal surface.

The focal surface may degenerate into a curve or developable, or it may break up into two surfaces either or both of which may be a curve or developable. If all the rays of a congruence intersect a curve that curve is a "degenerate" sheet of the focal surface, and reciprocally if all rays lie on tangent planes to a developable, the latter is a sheet of the focal surface. When the rays intersect two curves or the same curve twice the focal surface consists of the curve or curves, and the reciprocal theorem may be easily stated.]

For instance, the degeneration which has been just mentioned of necessity takes place when the congruence is algebraically of the first order. In this case, since through each point only one ray of the congruence can in general be drawn, a point cannot be the intersection of two of the lines unless it be a point through which an infinity of the lines can be drawn; and if the locus of points of intersection were a surface, every point of the surface would be a singular point, which is absurd. The locus is therefore a curve. If it be a proper curve, it must by definition be such that the cone standing on it, whose vertex is an arbitrary point, shall have one and but one apparent double line. This is the case when the curve is a twisted cubic, and there is no higher curve which has only one apparent double point. The only congruence then, of the first order, consisting of a system of lines meeting a proper curve twice, is when the curve is a twisted cubic. We might, however, have a congruence of lines meeting two directing curves, and if these curves be of the orders m, m', and have α common points, the order of the congruence will be $mm' - \alpha$. The only algebraic congruence of the first order of this kind is when the directing lines are a curve of the n^{th} order, and a right line meeting it $n - 1$ times.

[457*a*. *Surfaces connected with a congruence.* There are certain surfaces associated with a congruence to which attention may be directed: The *limit surface* is the two-sheeted locus of the limit points; the *focal surface* (Art. 457)

is the two-sheeted locus of the foci or envelope of the focal planes; the *middle surface* is the locus of the "middle point" which lies half-way between the limit points and as we have seen (Art. 457) coincides with the point half-way between the foci. The envelope of the plane drawn through the middle point perpendicular to the ray is called the *middle envelope;* and the two-sheeted envelope of the principal planes may be termed the *limit envelope.* The coordinates of the points on the first three surfaces, and of the tangent planes to the last two can at once be expressed in terms of the parameters p, q by the equations already given for the limit points and foci and for the values of t which determine the values of $dp : dq$ corresponding to the principal planes.

In addition to these unique surfaces, we have the two families of *developables* already defined, and the *principal surfaces* may also be noticed. These are two singly infinite families of ruled surfaces generated by rays of the congruence which are chosen so that the shortest distance between a generator and the consecutive generator intersects the former in a limit point, i.e. the limit points describe lines of striction (Art. 562). Each principal plane touches a principal surface at the corresponding limit point.

In general the developables meet the focal surfaces in two conjugate systems of curves. This follows at once from the definition of conjugate directions (Art. 268), since all the rays lie in tangent planes to each sheet of the focal surface.

The two focal points (but not the focal planes) will coincide on all rays touching the curve of intersection of the focal sheets. But they may coincide on *every* ray and in this case the two families of developables coalesce into one, the focal planes also coincide, and from the last paragraph or directly we see that the rays are tangents to one system of asymptotic lines on the focal surface. Such a congruence is said to be *parabolic.* A congruence or a portion thereof is said to be *hyperbolic* when the focal points on the rays are real and distinct; and *elliptic* when the rays though real have imaginary focal points. This classification is evidently determined

by the sign of the discriminant of either of the quadratic equations in Art. 457.*

As an example consider the congruence defined by the equations

$$l = by, \quad m = ax, \quad n = \sqrt{1 - a^2x^2 - b^2y^2}$$

the plane $z=0$ being the surface of reference. Clearly all real rays emanate from points within the ellipse $a^2x^2 + b^2y^2 = 1$. It will be found that the discriminant of the quadratic determining the values of $dx : dy$ for which two consecutive rays intersect is

$$\Delta \equiv a^2b^2(a + b)^2x^2y^2 + 4ab(1 - a^2x^2 - b^2y^2).$$

We leave it to the reader to show that if ab is positive the real portion of the congruence is hyperbolic; and that if ab is negative the equation $\Delta=0$ represents a curve lying within the ellipse, and dividing it into two regions for the outer of which the corresponding portion of the congruence is hyperbolic and for the inner elliptic, while the equation $\Delta=0$ corresponds to the curve of intersection of the focal sheets.

Ex. 1. The coordinates of a point on the middle surface are expressed in terms of two parameters p and q by the equations

$$\xi = x + lt, \quad \eta = y + mt, \quad \zeta = z + nt$$

$$\text{where } t = -\frac{ag + ec - f(b + b')}{2(eg - f^2)}$$

and x, y, z, the point where the ray meets the surface of reference, are known functions of p and q.

Ex. 2. The condition that the surface of reference may be the middle surface is

$$ag + ec - f(b + b') = 0.$$

Ex. 3. If the surface of reference be the middle surface the coordinates of a point ξ, η, ζ, on the middle envelope are given in terms of p and q by the equations

$$l\xi + m\eta + n\zeta = w$$

$$l_1\xi + m_1\eta + n_1\zeta = \frac{dw}{dp}$$

$$l_2\xi + m_2\eta + n_2\zeta = \frac{dw}{dq}$$

$$\text{where } w = lx + my + nz.]$$

[457*b*. *Normal Congruences.* A normal congruence is one whose rays are cut orthogonally by a singly infinite system of surfaces, which are therefore capable of being represented implicitly or explicitly by equations of the form

$$f(x, y, z) = \text{constant},$$

where the constant varies from surface to surface.

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* For the geometrical justification of the terms used see Sannia, *Geometria differenziale delle congruenze rettilinee* (Math. Ann., 58, 1910).

The preceding definition is, however, superfluous, for it has long been known that *if a congruence is normal to a single surface it is normal to a singly infinite system of surfaces.* Let the single surface ($S=0$) to which all the rays are normal be taken as the surface of reference. Any point (Q, ξ, η, ζ) on a ray through (P, x, y, z) is defined by

$$\xi = x + lt,\ \eta = y + mt,\ \zeta = z + nt,$$

where t is the distance PQ.

Hence $ld\xi + md\eta + nd\zeta = dt$, since $ldx + mdy + ndz = 0$. And if t is constant $ld\xi + md\eta + nd\zeta = 0$; that is, the surface locus of Q, as P moves along $S=0$, is normal to the rays of the congruence. Through any point in space we can describe one of these normal surfaces, by finding the locus of points equidistant from $S=0$; and so these surfaces form a singly infinite system, and the proposition stated has been proved.

The direction-cosines l, m, n may be supposed to be given as functions of the current coordinates ξ, η, ζ, of any point in space. If $ld\xi + md\eta + nd\zeta$ is a perfect differential dt, it is clear that the congruence thus defined is normal to the surfaces $t =$ constant. From what precedes it follows that the converse is also true, and so, as Hamilton pointed out, *the necessary and sufficient condition that a congruence be normal is that $ld\xi + md\eta + nd\zeta$ should be a perfect differential of a function of three independent coordinates.* (Cf. Ex. 1, Art. 455*p*.)

Now let the surface of reference be chosen arbitrarily, and let l, m, n be a normal congruence defined by two parameters p and q. We then have as before

$$ld\xi + md\eta + nd\zeta = ldx + mdy + ndz + dt = Pdp + Qdq + dt.$$

The condition requires that $ld\xi + md\eta + nd\zeta$, when expressed in any three independent parameters, should be a perfect differential, and therefore since

$$P = lx_1 + my_1 + nz_1,\ Q = lx_2 + my_2 + nz_2,$$

we must have, because $Pdp + Qdq$ is a perfect differential,

$$\frac{d}{dq}(lx_1 + my_1 + nz_1) = \frac{d}{dp}(lx_2 + my_2 + nz_2)$$

and therefore $l_1x_2 + m_1y_2 + n_1z_2 = l_2x_1 + m_2y_1 + n_2z_1$, i.e. *the*

necessary and sufficient condition that a congruence, expressed by any two parameters with any surface of reference, may be normal, is

$$b = b'.$$

Now if we compare the quadratics for the limit points (Art. 456) and for the focal distances (Art. 457), we see that they are of the form

$$Ar^2 + Br + ac - \tfrac{1}{4}(b + b')^2 = 0, \quad A\rho^2 + B\rho + ac - bb' = 0.$$

Thus the condition $b = b'$ is equivalent to the condition that these equations have the same roots. Hence *a normal congruence is one whose focal points coincide with the limit points.* In this case both coincide with the centres of curvature of the orthogonal surfaces, and the focal surface, limit surface, and limit envelope coincide with the surface of centres. The focal planes are therefore at right angles.

Again, if we use the expression given in Ex. 2, Art. 457, for the direction-cosines of the normal to a focal plane, we shall find that the cosine of the angle (θ) between the two focal planes is

$$\frac{edp_1dp_2 + f(dp_1dq_2 + dp_2dq_1) + gdq_1dq_2}{\sqrt{(edp_1^2 + 2fdp_1dq_1 + gdq_1^2)(edp_2^2 + 2fdp_2dq_2 + gdq_2^2)}}$$

where $dp_1 : dq_1$ and $dp_2 : dq_2$ are roots of the quadratic in $dp : dq$ given in Art. 457. It will be found that when $\cos\theta$ is expressed in terms of a, b, b', c, e, f, g, the numerator of the fraction after reduction is $(eg - f^2)(b - b')$, and hence *the necessary and sufficient condition that a congruence be normal is that the focal planes cut at right angles.*

Ex. 1. By taking the middle surface for the surface of reference and making $dp = 0$, $dq = 0$ correspond to the focal planes it can be proved that for any congruence the sine of the angle between the focal planes is $\frac{\rho}{r}$, where r is the distance between the focal points and ρ that between the limit points.

Ex. 2. The congruence formed by common tangent lines to two confocals and, in particular, that formed by lines meeting each of two focal conics, are normal (see Art. 176).

The following theorem, due to Malus and Dupin,* is important in geometrical optics: *A normal congruence remains normal after refraction or reflexion at any surface.* The refracting surface may be taken as the surface of reference, and if μ be the constant index of refraction †

$$\sin i = \mu \sin i'$$

where i and i' are the angles between the normal and the incident and refracted rays. Hence if the rays are l, m, n, and l', m', n', we have easily

$$l = \lambda u + \mu l'$$
$$m = \lambda v + \mu m'$$
$$n = \lambda w + \mu n'$$

where u, v, w are the direction-cosines of the normal to the surface of reference. Hence, along the surface of reference

$$l dx + m dy + n dz = \mu\,(l' dx + m' dy + n' dz).$$

Thus, if $l dx + m dy + n dz$ is a perfect differential of a function of p and q the same is true of $l' dx + m' dy + n' dz$, and therefore (p. 67) if the first congruence is normal so is the second.

There is a simple geometrical relation between the normal surfaces of the two congruences. Let Q (ξ, η, ζ) be a point on the incident ray, P (x, y, z) the point where this ray meets the refracting surface, and Q' (ξ', η', ζ') a point on the refracted ray. If $QP = \rho$, $PQ' = \rho'$, then it is easy to prove

$$l d\xi + m d\eta + n d\zeta - \mu(l' d\xi' + m' d\eta' + n' d\zeta')$$
$$= \lambda(u dx + v dy + w dz) + d\rho + \mu d\rho',$$

and therefore $l' d\xi' + m' d\eta' + n' d\zeta' = 0$, provided $d\rho + \mu d\rho' = 0$. Hence the refracted rays are normal to the surfaces derived from any surface normal to the incident rays by taking the loci of points for which $\rho + \mu\rho'$ is constant.

Ex. 3. Prove that a surface may be found such that any normal congruence will be refracted through it with any index so that (*a*) the emergent rays will pass through a given point or more generally (*b*) coincide with the rays of any other normal congruence.

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* Malus proved that the rays of a star when reflected from a surface are normal to some surface; Dupin extended the theorem as here given.

† For reflexion $\mu = -1$.

Thus any normal congruence may be illustrated—theoretically at all events—by the refraction through some surface of a system of rays of light proceeding from a fixed point chosen arbitrarily.

When a surface is deformed a congruence may be said to be deformed with it if the ray at a point on a surface does not alter its angular relations to the linear elements of the surface in the neighbourhood of the point. Beltrami first showed that *a normal congruence remains normal if deformed with any surface.*

Take the surface as that of reference and suppose that the parametric lines correspond. Then expressing the condition (Art. 390) that E and G and the angles between the ray and the parametric lines are unaltered we find that $lx_1+my_1+nz_1$ and $lx_2+my_2+nz_2$ are unaltered, and therefore $ldx+mdy+ndz$, since it $=(lx_1+my_1+nz_1)\ dp+(lx_2+my_2+nz_2)\ dq$, continues to be a perfect differential.

We may here mention an associated theorem which is a particular case of a general theorem due to Ribaucour (Art. 486d). *If tangent planes be drawn through the rays of a rectilinear normal congruence to any surface the congruence remains normal if the surface be deformed in any manner carrying the rays in its tangent planes* in the way just indicated. As the general proof will be given later we do not insert it here.

The rays of *any* congruence are tangent lines to a singly infinite system of curves on either sheet of a focal surface, and the other sheet is the envelope of the osculating planes of these curves. If the congruence is normal these curves are geodesics on the focal surface, which of course is the surface of centres (Art. 309). Thus *the rays of every normal congruence are tangent lines to a singly infinite family of geodesics on some surface.* Conversely *the tangent lines to a singly infinite family of geodesics on a surface form a normal congruence*; for one focal plane of a ray is the tangent plane at the point where the ray touches the surface, while the second focal plane is the osculating plane to the geodesic at the

same point, and since these planes are at right angles (Art. 308) the congruence is normal (p. 68). Since the envelope of a singly infinite family of curves on a surface is another curve (real or imaginary) on the same surface, every normal congruence consists of the rectilinear prolongations of geodesics touching a fixed curve on some surface. The envelope may of course reduce to a point or to a group of points. Thus a normal congruence may be illustrated by stretching a string over a fixed surface as the following example shows :—

Ex. 4. If a thread be passed tightly round any part of a curve on a surface, or attached to a fixed point thereon, and stretched over the surface to a point in space, the possible positions of its rectilinear portions form a normal congruence, and any point of the thread describes a normal surface.

Ex. 5. The common tangent lines to two confocal quadrics form a normal congruence, whose rays continued geodesically on either surface touch their common line of curvature, which reduces to an umbilic when one of the confocals reduces to a focal conic. (Cf. Art. 405.)

Ex. 6. Using elliptic coordinates (Art. 421*a*), the surfaces normal to the congruence of Ex. 3 are the family

$$\int f(\lambda)\, d\lambda + \int f(\mu)\, d\mu + \int f(\nu)\, d\nu = \text{constant}$$

where
$$f(t) = \frac{\sqrt{(\mu_0 - t)(\lambda_0 - t)}}{\sqrt{(\alpha - t)(\beta - t)(\gamma - t)}}.$$

Use the expression (Art. 421*a*) for twice the element of length of a ray, namely

$$f(\lambda)\, d\lambda + f(\mu)\, d\mu + f(\nu)\, d\nu.$$

Ex. 7. If the curves on a surface orthogonal to the family q = constant, be geodesics, their tangent lines are normal to the family of surfaces

$$\int \sqrt{G}dq + \rho = \text{constant},$$

where ρ is the length of a tangent line (Art. 396*a*, Ex. 1).]

[457*c*. *Directed Normal Congruences.* If a sheet of the focal surface degenerates into a curve (see end of Art. 457), all the rays meet this curve and the congruence may be described as *directed.* We may apply the term *singly-directed* if each ray meets a fixed curve once only, and the term *doubly-directed* if all the rays meet each of two fixed curves once, or the same curve twice. A doubly-directed congruence is determined if we are given both the directing curves.

Suppose that all the rays of a congruence meet a curve C_1. Then one focal plane $(L=0)$ on any ray is the plane containing the ray and the tangent line to C_1 at the point P where the ray meets the curve. The second focal plane $(M=0)$ is clearly the tangent plane through the ray to the cone of rays that pass through P and are tangent to the second focal sheet (or which intersect the second directing curve if there is one). In the case of a normal congruence the planes L and M are mutually perpendicular (p. 68), and so the tangent plane along any generator of the cone is perpendicular to the plane containing the generator and a fixed line through P. The sections of the cone by planes perpendicular to that fixed line must therefore be circles, because the tangent at any point of a section is perpendicular to the line joining the point to a fixed point in its plane. Thus the cone is one of revolution, and its axis is the tangent line to C_1. Conversely if every cone is one of revolution round the tangent line, the congruence is normal. We have proved then that *the necessary and sufficient condition that a directed congruence be normal is that the tangent cone from any point of the directing curve to the second sheet of the focal surface should be a cone of revolution round the tangent line to the directing curve at the vertex.*

If the rays also meet a second curve (C_2) this forms the second sheet of the focal surface, and if the congruence be normal, any cone passing through either curve and having its vertex on the other is a cone of revolution. Both the curves must therefore be conics since (Art. 200) it is not otherwise possible to draw more than four quadric cones through the same curve. Hence (Art. 184) we have the remarkable theorem: *The necessary and sufficient condition that a doubly-directed congruence should be normal is that the directing curves should be focal conics of a confocal system of quadrics.*]

[457*d*. *Cyclides of Dupin.* We shall now consider the surfaces normal to a doubly-directed congruence. Let P be

a point on a directing curve of a directed normal congruence. A sphere with P as centre intersects the tangent cone from P to the second focal sheet in a circle which is clearly a line of curvature on a normal surface. Thus if the normals to a surface form a directed congruence the lines of curvature of one system are circles, and it follows that *if the normals to a surface form a doubly-directed congruence, the lines of curvature of both systems are circles.* Such surfaces are known as *Cyclides of Dupin.**

Conversely it may be seen by using Lancret's theorem (Art. 312) that if on a surface the lines of curvature of one system are circles the centres of curvature for all the points on a given circle reduce to a single point symmetrically placed with regard to the circle, and therefore since the circles form a singly infinite system, the corresponding sheet of the surface of centres is a curve. We infer that *if the lines of curvature of both systems are circles the normals to the surface form a doubly-directed congruence.*

Corresponding to any quadric we have three systems of parallel cyclides of Dupin, namely, those whose normals intersect a pair of focal conics. Only one system is real, and its equations may be expressed very simply in elliptic coordinates by the method suggested in Ex. 6, Art. 457*b*. In this case the two quadrics reduce to the focal conics and then (Art. 421*a*) $\lambda_0=\gamma,\ \mu_0=\beta$. The equation

$$\int f(\lambda)\, d\lambda + \int f(\mu)\, d\mu + \int f(\nu)\, d\nu = \text{constant}$$

then becomes

$$\sqrt{a-\lambda} + \sqrt{a-\mu} + \sqrt{a-\nu} = \text{constant},$$

or using the notation of Art. 160

$$a' \pm a'' \pm a''' = t$$

where t is a constant.

If this equation be rationalized it will be found (Art. 160) that the cyclide referred to the axes of the confocal system is the surface of the fourth order

* See Dupin's *Applications de géométrie et de méchanique* (1822).

$$(x^2+y^2+z^2-h^2-k^2+t^2)^2$$
$$=4(t^2-h^2)y^2+4(t^2-k^2)z^2+4(tx+hk)^2.$$

Referring to Art. 421*b*, we see that the equations $r_1=t+\sqrt{a-\gamma}$, $r_2=t+\sqrt{a-\gamma}$, $s_1=t-\sqrt{a-\beta}$, $s_2=t-\sqrt{a-\beta}$ represent a cyclide of the system. From this we derive the following mechanical construction: *If a string of constant length be attached at one end to the focus of a conic and stretched tight over the conic, the locus of its extremity is a portion of a Dupin cyclide.* This is a special case of the theorem of Ex. 4, Art. 457*b*.

The cyclide may be defined in various ways as an envelope. Consider the two circles of curvature at a point P; these lie respectively on two spheres whose centres are on the focal conics, and since the line joining the centres is the normal to the cyclide at P it passes through a point of intersection of the spheres; hence the two spheres touch at P. The spheres also touch the cyclide which is thus the envelope of a one parameter system of spheres whose centres are on the focal conics, each sphere of one system touching all those of the other. But since three spheres are enough to determine an envelope, the cyclide is definable as *the envelope of a sphere touching three fixed spheres* (see also Art. 567).

Conversely every such envelope is a cyclide of Dupin. For if we invert from a point common to the three spheres, the inverse envelope is the envelope of a sphere touching three fixed planes and is therefore a right cone. The lines of curvature of a right cone are the generators and the circular sections, and therefore since lines of curvature are preserved in inversion (Art. 384), those on the original envelope are circles.

Ex. 1. Prove that the equation of the envelope of a sphere touching three fixed spheres $S_1=0$, $S_2=0$, $S_3=0$ may be written

$$t_{23}\sqrt{S_1}+t_{31}\sqrt{S_2}+t_{12}\sqrt{S_3}=0$$
$$\text{where } S_1\equiv(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r_1^2$$
$$\text{and } t_{23}^2=(x_2-x_3)^2+(y_2-y_3)^2+(z_2-z_3)^2-(r_1-r_2)^2.$$

Ex. 2. Any cyclide of Dupin inverts into another.

Ex. 3. If the directing conics are parabolas in perpendicular planes each passing through the focus of the other, the corresponding cyclides are of the third order.

Ex. 4. All circles drawn through a fixed point and cutting orthogonally a given Dupin-cyclide, intersect two fixed sphero-curves.]

[457*e*. *Ribaucour's Isotropic Congruences.* We have seen (Art. 456) that the points where a ray meets the shortest

distance from a consecutive ray lie between the limit points L_1 and L_2. But all these points may coincide and a ray will then meet all shortest distances at the middle point M. A congruence all of whose rays possess this property is said to be *isotropic*.* The middle surface coincides with the limit surface and is real, but it may be shown that the focal surface is imaginary.

Since the points where a ray meets its shortest distance from a consecutive ray is given (Art. 456) by

$$-r = \frac{adp^2 + (b+b')\,dpdq + cdq^2}{edp^2 + 2fdpdq + gdq^2},$$

and since in the present case all these values are equal, *the necessary and sufficient condition that a congruence be isotropic is*

$$\frac{a}{e} = \frac{b+b'}{2f} = \frac{c}{g}$$

each of these with sign changed being the distance of the middle point from the point where the ray meets the surface of reference. Hence if the middle surface be taken for the surface of reference we have $a = b + b' = c = 0$ or what amounts to the same, for all variations on the middle surface $dxdl + dydm + dzdn = 0$.

We notice that an isotropic congruence is uniquely defined by the property that *the lines of striction of all its ruled surfaces lie on a surface* (the middle surface).

Ex. 1. Through each point of the plane $z = 0$ a ray l, m, n is drawn, where

$$l = ky, \quad m = -kx, \quad n = \sqrt{1 - k^2(x^2 + y^2)}.$$

Prove that the congruence so formed is isotropic, that $z = 0$ is the middle surface and that the middle envelope (Art. 457a) reduces to a point.

Ex. 2. The only normal isotropic congruence is a system of rays passing through a point.

Ex. 3. If the distance between corresponding points on two applicable surfaces be constant the joining lines form an isotropic congruence, whose middle surface is the locus of the middle point of these lines.

Conversely if the same constant length be measured in each direction

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* These congruences were fully investigated by Ribaucour in connexion with minimal surfaces (see Art. 457f) in his *Étude des Élassoïdes ou Surfaces à Courbure moyenne nulle*, "Mémoires Couronnés par L'Academie Royale de Belgique," XLIV, 1882.

along a ray from the middle surfaces of an isotropic congruence the loci of the extremities are applicable surfaces.

Ex. 4. The spherical representation of a congruence, or of a portion thereof, is sometimes defined as the arrangement of points in which radii through the centre of a fixed sphere and parallel to rays of the congruence meet the surface of the sphere. Prove that if the congruence be isotropic corresponding linear elements of any curve C on the middle surface and of the spherical representation of the rays through points on C are orthogonal.

Ribaucour gives the following method of generating isotropic congruences. Let p and q be isometric parameters (Art. 392a) on any sphere, and let $d\sigma$ be the element of an arc of the sphere; then $d\sigma^2 = \lambda^2\ (dp^2 + dq^2)$ where λ is a function of p and q. At each point Π of the sphere draw a tangent line to one of the parametric curves, and measure on this line a distance $\Pi P = \lambda$. Then the ray through P drawn parallel to the normal (l, m, n) to the sphere at Π generates an isotropic congruence.

Ex. 5. For example the congruence of Ex. 1 may be defined by taking, along the tangent line to the parallel of latitude at Π, a length ΠP proportional to the cosine of the latitude, and drawing through P a ray parallel to the radius of the sphere at Π. (Cf. Art. 392a, p. 419.)

Conversely every isotropic congruence has the relation just mentioned to any sphere. There is no difficulty in proving the original proposition, but we shall indicate a method of proving the converse. Let Π be the spherical representation of a ray R of an isotropic congruence, i.e. Π is the point where a fixed sphere is met by the radius parallel to the ray. Let the tangent plane to the sphere at Π cut the ray in P. Then the system of lines $P\Pi$ envelopes a singly infinite system of curves on the sphere, and we have to show that these curves with their orthogonal trajectories on the sphere form an isothermal system, i.e. they can be represented by the parametric equations $u =$ constant, $v =$ constant, so that

$$d\sigma^2 = E\ (du^2 + dv^2)$$

where $d\sigma$ is an element of the spherical arc. Furthermore $P\Pi$ is proportional to $\sqrt{E}$.

The ray being l, m, n, and the radius of the sphere unity, the coordinate of Π are lmn. Let $\Pi P = r$, and let ΠP be tangent at Π to the curve $q =$ constant. Then if we take the curves $p =$ constant to represent the orthogonal trajectories of $q =$ constant we have

$$d\sigma^2 = edp^2 + gdq^2, \text{ and } f = 0.$$

The coordinates of P are x, y, z where

$$x = l + \mu l_1,\ y = m + \mu m_1,\ z = n + \mu n_1$$

where $\mu = \frac{r}{\sqrt{e}}$.

The conditions for an isotropic congruence, since $f = 0$, are

$$b + b' = 0,\quad \frac{a}{e} = \frac{c}{g}$$

and these will be found to yield the equations

$$\frac{d\mu}{dq} = 0,\quad \frac{2}{\mu}\frac{d\mu}{dp} = \frac{1}{g}\frac{dq}{dp} - \frac{1}{e}\frac{de}{dp}$$

from which we infer that μ is a function of p alone, and that $\frac{\mu^2 e}{g}$ is a function of q alone. Thus $\frac{e}{g}$ is of the form $\frac{\phi(p)}{\psi(q)}$, and we find

$$d\sigma^2 = E\,(du^2 + dv^2)$$

where u is a function of p only, and v of q only, and the parametric lines are therefore unaltered. Hence the parametric lines are isothermal. Again the preceding equations show that if $e = g$ then μ is constant, and therefore $\frac{r}{\sqrt{E}}$ is constant.

Taking the middle surface for the surface of reference we have $a = 0$, $b + b' = 0$, $c = 0$, and the values of $dp : dq$ corresponding to the focal points (Art. 457) are given by

$$edp^2 + 2fdpdq + gdq^2 = 0.$$

Since the left hand is $dl^2 + dm^2 + dn^2$ (Art. 456), the focal points, planes and surface of the congruence are imaginary. Moreover (Ex. 2, Art. 457), it is easily seen that if a focal plane be written in the form

$$\alpha(\xi - x) + \beta(\eta - y) + \gamma(\zeta - z) = 0$$

we must have $\alpha^2 + \beta^2 + \gamma^2 = 0$. Hence (Art. 211) the focal planes of an isotropic congruence touch the imaginary circle at infinity, and therefore *the focal surface is a developable containing the imaginary circle at infinity.** The converse

* The nature of a congruence whose focal sheets are developable is easily understood by comparing it with the reciprocal case of a doubly-directed congruence (Art. 457c). The rays are the intersections of pairs of tangent planes.

may easily be proved. A developable of this type is sometimes described as isotropic. Thus *an isotropic congruence may be defined as one whose focal surface is an isotropic developable,* or, more generally, two isotropic developables.* The congruence is real if the developables are conjugate imaginaries, otherwise it is imaginary.

Ex. 6. The generators of one system of a family of quadrics passing through a common curve form a doubly directed congruence determined by that curve. Reciprocally the generators of quadrics touched by a common developable form a congruence whose focal surface coincides with that developable.

Hence prove Ribaucour's theorem : *The generators of either system of a system of confocal hyperboloids of one sheet form an isotropic congruence.*

It may be noticed that the generators of one system are the reflexions of the generators of the other system with regard to any confocal quadric of different species. Ribaucour proves also the converse, that if two congruences which are mutual reflexions with regard to a surface are isotropic, the surface is a quadric and the rays of the congruences are generators of quadrics confocal thereto.]

[457*f*. One of the most interesting properties of the congruence considered is this (Ribaucour): *The middle envelope of an isotropic congruence is a minimal surface, that is, its mean curvature vanishes* (Art. 295, Ex. 1).

The demonstration depends on certain lemmas :—

(1) If the direction-cosines of the normal at a point Q (ξ, η, ζ) on a surface $S=0$ are l, m, n, expressed in terms of two parameters p, q, the condition for a minimal surface is

$$eD'' + gD - 2fD' = 0$$

where $e = l_1^2+m_1^2+n_1^2$, $f = l_1l_2 + m_1m_2 + n_1n_2$, $g = l_2^2 + m_2^2 + n_2^2$, $D = l_1\xi_1 + m_1\eta_1 + n_1\zeta_1$, $D' = l_1\xi_2 + m_1\eta_2 + n_1\zeta_2 = l_2\xi_1 + m_2\eta_1 + n_2\zeta_1$ and $D'' = l_2\xi_2 + m_2\eta_2 + n_2\zeta_2$, where the suffixes 1 and 2 denote differentiation with regard to p and q respectively.

Let Q' (ξ', η', ζ') be a centre of curvature on $S=0$ corresponding to Q, and let r be the principal radius QQ'. We have three equations

$$\xi' = \xi + rl, \quad \eta' = \eta + rm, \quad \zeta' = \zeta + rn$$

and therefore $d\xi' = d\xi + rdl + ldr$, with two similar equations.

But if $\xi\eta\zeta$ be supposed to move along a line of curvature $d\xi' = ldr$, etc., since dr is an element of the arc along which ξ' moves, and therefore

$$d\xi + rdl = 0, \text{ etc.}$$

Using $d\xi = \xi_1 dp + \eta_1 dq$, $dl = l_1dp + l_2dq$, etc., we find that the value of $dp : dq$ for the principal directions are given by

* This is in fact the definition from which Ribaucour starts.

$$Ddp + D'dq - r(edp + fdq) = 0$$
$$D'dp + D''dq - r(fdp + gdq) = 0$$

and the principal radii are given by the quadratic

$$(eg - f^2)r^2 - (eD'' + gD - 2fD')r + DD'' - D'^2 = 0.$$

Consequently the condition for a minimal surface is

$$eD'' + gD - 2fD' = 0$$

and if $e = 0$, $g = 0$ the condition is $D' = 0$ or $l_1\xi_2 + m_1\eta_2 + n_1\zeta_2 = 0$.

(2) Consider an isotropic congruence, and suppose its middle surface $M = 0$ is the surface of reference. Let P (x, y, z) be the middle point of a ray whose direction-cosines are l, m, n; then x, y, z, l, m, n are functions of two parameters p, q.

Let Q (ξ, η, ζ) be the point of contact of the middle plane through P (i.e. the plane perpendicular to the ray) with the middle envelope $S = 0$. Then ξ, η, ζ are functions of p and q.

The conditions for an isotropic congruence are

$$a = l_1x_1 + m_1y_1 + n_1z_1 = 0 \qquad \text{(i)}$$
$$b + b' = l_1x_2 + m_1y_2 + n_1z_2 + l_2x_1 + m_2y_1 + n_2z_1 = 0 \qquad \text{(ii)}$$
$$c = l_2x_2 + m_2y_2 + n_2z_2 = 0 \qquad \text{(iii)}$$

We may choose the parameters p, q so that the parametric lines p = constant, q = constant correspond to the "minimal lines" in the spherical representation of the congruence (Art. 392a). This implies $dl^2 + dm^2 + dn^2 = 2fdpdq$, and $e = g = 0$, i.e.

$$l_1^2 + m_1^2 + n_1^2 = 0 \qquad \text{(iv)}$$
$$l_2^2 + m_2^2 + n_2^2 = 0 \qquad \text{(v)}$$
$$l^2 + m^2 + n^2 = 1 \qquad \text{(vi)}$$

Also since P and Q both lie on the middle plane

$$lx + my + nz = l\xi + m\eta + n\zeta \qquad \text{(vii)}$$

and, because the direction-cosines of the normal at Q to $S = 0$ are l, m, n, we have

$$l\xi_1 + m\eta_1 + n\zeta_1 = 0 \qquad \text{(viii)}$$

and

$$l\xi_2 + m\eta_2 + n\zeta_2 = 0 \qquad \text{(ix)}$$

Using these equations we can prove the following relations, where

$$f = l_1l_2 + m_1m_2 + n_1n_2$$
$$l_{11} = \frac{1}{f}\frac{df}{dp}\,l_1, \text{ with two similar equations} \qquad (A)$$

and

$$l_{12} = -lf, \text{ with two similar equations} \qquad (B)$$

Proof of (A). Differentiating both sides of equations (iv) and (vi) with regard to p, $\Sigma l_1l_{11} = 0$ and $\Sigma l_1l = 0$, and by a second differentiation of the latter, using (iv), $\Sigma ll_{11} = 0$. Hence since $\Sigma l_1^2 = 0$, and $\Sigma ll_1 = 0$, we find

$$l_{11} = \lambda l_1, \quad m_{11} = \lambda m_1, \quad n_{11} = \lambda n_1,$$

and therefore

$$\Sigma l_2l_{11} = \lambda f.$$

But

$$\frac{df}{dp} = \Sigma l_2l_{11} \text{ since } \Sigma l_1l_{12} = 0 \text{ by (iv)}$$

therefore

$$l_{11} = \frac{1}{f}\frac{df}{dp}\,l_1,$$

Proof of (B). The equations (iv) and (v) give $\Sigma l_1 l_{12} = 0$, $\Sigma l_2 l_{12} = 0$, and since from (vi) $\Sigma l_1 l = 0$, $\Sigma l_2 l = 0$, we have

$$l_{12} = \mu l, \quad m_{12} = \mu m, \quad n_{12} = \mu n,$$

and by using (vi) $\mu = \Sigma l l_{12} = -\Sigma l_1 l_2 = -f$, which proves the formula (B).

(3) We can now prove

$$lx_{12} + my_{12} + nz_{12} = l_1\xi_2 + m_1\eta_2 + n_1\zeta_2$$

and the condition that $S = 0$ may be a minimal surface is therefore

$$lx_{12} + my_{12} + nz_{12} = 0.$$

Differentiating both sides of equation (vii), viz. $\Sigma lx = \Sigma l\xi$, with regard to p and q successively and using equations (ii) and (viii)

$$\Sigma lx_{12} + \Sigma l_{12}x = \Sigma l_1\xi_2 + \Sigma l_{12}\xi.$$

But since $\Sigma lx = \Sigma l\xi$ the equation (B) gives $\Sigma l_{12}x = \Sigma l_{12}\xi$ and consequently

$$lx_{12} + my_{12} + nz_{12} = l_1\xi_2 + m_1\eta_2 + n_1\zeta_2.$$

(4) To prove $\Sigma lx_{12} + my_{12} + nz_{12} = 0$, use the identity

$$2(lx_{12} + my_{12} + nz_{12}) = \frac{d}{dq}(lx_1 + my_1 + nz_1) + \frac{d}{dp}(lx_2 + my_2 + nz_2)$$

which follows from equation (ii).

Now by (B), $f\Sigma lx_1 = -\Sigma l_{12}x_1 = \Sigma l_1 x_{12}$, by (i),

$$= \frac{d}{dp}(\Sigma l_1 x_2) - \Sigma l_{11}x_2 = \frac{db}{dp} - \frac{b}{f}\frac{df}{dp}, \text{ by } (A).$$

Hence $lx_1 + my_1 + nz_1 = \dfrac{d}{dp}\left(\dfrac{b}{f}\right)$, and similarly, $lx_2 + my_2 + nz_2 = \dfrac{d}{dq}\left(\dfrac{b'}{f}\right)$

and therefore, since $b + b' = 0$, it follows that

$$lx_{12} + my_{12} + nz_{12} = 0$$

and the proposition of this article is proved.]

SECTION III.—RULED SURFACES.

458. On account of the importance of ruled surfaces, we add some further details as to this family of surfaces.

The tangent plane at any point on a generator evidently contains that generator, which is one of the inflexional tangents (Art. 265) at that point. Each different point on the generator has a different tangent plane (Art. 110), and the following construction gives the tangent plane and the second inflexional tangent. We know that through a given point can be drawn a line intersecting two given lines; namely, the intersection of the planes joining the given point to the given lines. Now consider three consecutive generators, and through any point A on one draw a line meeting the other two. This line, passing through three consecutive

points on the surface, will be the second inflexional tangent at A, and therefore the plane of this line and the generator at A is the tangent plane at A. In this construction it is supposed that two consecutive generators do not intersect, which ordinarily they will not do. There may be on the surface, however, singular generators which are intersected by a consecutive generator, and in this case the plane containing the two consecutive generators is a tangent plane at every point on the generator. In special cases also two consecutive generators may coincide, in which case the generator is a double line on the surface.

[Any surface may be regarded as generated by the motion of a curve

$$f_1(x, y, z, w, t) = 0, f_2(x, y, z, w, t) = 0$$

where t is a variable parameter. Eliminating dt from the two equations of the type

$$\frac{df_1}{dx}dx + \frac{df_1}{dy}dy + \frac{df_1}{dz}dz + \frac{df_1}{dw}dw + \frac{df_1}{dt}dt = 0$$

we find

$$Xdx + Ydy + Zdz + Wdw = 0$$

and it is clear that $XYZW$ are coordinates of the tangent plane at x, y, z, w.

In the case of a ruled surface $f_1 = 0$ and $f_2 = 0$ may be taken to be planes

$$\alpha_1 x + \beta_1 y + \gamma_1 z + \delta_1 w = 0, \quad \alpha_2 x + \beta_2 y + \gamma_2 z + \delta_2 w = 0$$

and it will be found that the coordinates of the tangent plane are

$$\alpha_1 L_2 - \alpha_2 L_1, \beta_1 L_2 - \beta_2 L_1, \gamma_1 L_2 - \gamma_2 L_1, \delta_1 L_2 - \delta_2 L_1,$$

where L_1, L_2 are linear in x, y, z, w, and their coefficients are functions of t. There is thus a homographic correspondence between the points on a generator and the tangent planes at these points, i.e. if $T = 0$, $T' = 0$ be the tangent planes at $xyzw$ and $x'\,y'\,z'\,w'$, then $T + \lambda T' = 0$ is the tangent plane at the point $x + \lambda x', y + \lambda y', z + \lambda z', w + \lambda w'$. This is proved synthetically in the following article.]

459. *The anharmonic ratio of four tangent planes passing through a generator is equal to that of their four points of contact.* Let three fixed lines A, B, C be intersected by four transversals in points $aa'a''a'''$, $bb'b''b'''$, $cc'c''c'''$. Then the anharmonic ratio $\{bb'b''b'''\} = \{cc'c''c'''\}$, since either measures the ratio of the four planes drawn through A and the four transversals. In like manner $\{cc'c''c'''\} = \{aa'a''a'''\}$, either measuring the ratio of the four planes through B (see Art. 114). Now let the three fixed lines be three consecutive generators of the ruled surface, then, by the last article, the

transversals meet any of these generators A in four points, the tangent planes at which are the planes containing A and the transversals. And by this article it has been proved that the anharmonic ratio of the four planes is equal to that of the points where the transversals meet A.

[Ex. Prove that the generators of a ruled surface are cut equianharmonically by the other system of asymptotic lines.]

460. It is well known that a series of planes through any line and a series through it at right angles to the former constitute a system in involution, since the anharmonic ratio of any four is equal to that of their four conjugates. It follows then from the last article that the system formed by the points of contact of any plane, and of a plane at right angles to it, form a system in involution; or, in other words, the system of points where planes through any generator touch the surface, and where they are normal to the surface form a system in involution. The centre of the system is the point where the plane which touches the surface at infinity is normal to the surface; and, by the known properties of involution, the rectangle under the distances from this point of the points where any other plane touches and is normal, is constant.

461. *The normals to any ruled surface along any generator generate a hyperbolic paraboloid.* It is evident that they are all parallel to the same plane, namely, the plane perpendicular to the generator. We may speak of the anharmonic ratio of four lines parallel to the same plane, meaning thereby that of four parallels to them through any point. Now in this sense the anharmonic ratio of four normals is equal to that of the four corresponding tangent planes, which (Art. 459) is equal to that of their points of contact, which again (Art. 460) is equal to that of the points where the normals meet the generator. But a system of lines parallel to a given plane and meeting a given line generates a hyperbolic paraboloid, if the anharmonic ratio of any four is equal to that of the four

points where they meet the line. This proposition follows immediately from its converse, which we can easily establish.

The points where four generators of a hyperbolic paraboloid intersect a generator of the opposite kind are the points of contact of the four tangent planes which contain these generators, and therefore the anharmonic ratio of the four points is equal to that of the four planes. But the latter ratio is measured by the four lines in which these planes are intersected by a plane parallel to the four generators, and these intersections are lines parallel to these generators.

462. The central points of the involution (Art. 460) are, it is easy to see, the points where each generator is nearest the next consecutive; that is to say, the point where each generator is intersected by the shortest distance between it and its next consecutive. The locus of the points on the generators of a ruled surface, where each is closest to the next consecutive, is called the *line of striction* of the surface. It may be remarked, in order to correct a not unnatural mistake (see *Lacroix*, vol. III. p. 668), that the shortest distance between two consecutive generators is *not* an element of the line of striction. In fact, if Aa, Bb, Cc be three consecutive generators, ab the shortest distance between the two former, then $b'c$ the shortest distance between the second and third will in general meet Bb in a point b' distinct from b, and the element of the line of striction will be ab' and not ab.

Ex. 1. To find the line of striction of the hyperbolic paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z.$$

Any pair of generators may be expressed by the equations

$$\frac{x}{a} + \frac{y}{b} = \lambda z, \quad \frac{x}{a} - \frac{y}{b} = \frac{1}{\lambda},$$

$$\frac{x}{a} + \frac{y}{b} = \mu z, \quad \frac{x}{a} - \frac{y}{b} = \frac{1}{\mu}.$$

Both being parallel to the plane $\frac{x}{a} - \frac{y}{b}$, their shortest distance is perpendicular to this plane, and therefore lies in the plane

$$(a^2+b^2)\left\{\frac{x}{a}+\frac{y}{b}-\mu z\right\}+(a^2-b^2)\left\{\frac{x}{a}-\frac{y}{b}-\frac{1}{\mu}\right\},$$

which intersects the first generator in the point $z=\dfrac{a^2-b^2}{a^2+b^2}\dfrac{1}{\lambda\mu}$.

When the two generators approach to coincidence, we have for the coordinates of the point where either is intersected by their shortest distance

$$z=\frac{a^2-b^2}{a^2+b^2}\frac{1}{\lambda^2},\qquad \frac{x}{a}+\frac{y}{b}=\frac{a^2-b^2}{a^2+b^2}\frac{1}{\lambda},$$

and hence $$(a^2+b^2)\left(\frac{x}{a}+\frac{y}{b}\right)=(a^2-b^2)\left(\frac{x}{a}-\frac{y}{b}\right),\text{ or }\frac{x}{a^3}+\frac{y}{b^3}=0.$$

The line of striction is therefore the parabola in which this plane cuts the surface. The same surface considered as generated by the lines of the other system has another line of striction lying in the plane

$$\frac{x}{a^3}-\frac{y}{b^3}=0.$$

Ex. 2. To find the line of striction of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Ans. It is the intersection of the surface with

$$\frac{a^2A^2}{x^2}+\frac{b^2B^2}{y^2}=\frac{c^2C^2}{z^2},$$

where $$A=\frac{1}{b^2}+\frac{1}{c^2},\qquad B=\frac{1}{a^2}+\frac{1}{c^2},\qquad C=\frac{1}{b^2}-\frac{1}{a^2}.$$

[462*a*. The parametric method of Gauss may be conveniently applied to ruled surfaces by the use of a device analogous to that employed for congruences (Art. 456). Let x, y, z be the coordinates of any point (P) on any fixed director curve, $p=$ constant, lying on the surface. If $Q(\xi, \eta, \zeta)$ be any point on a generator through P we have

$$\xi=x+ql,\qquad \eta=y+qm,\qquad \zeta=z+qn$$

where $q=PQ$, and l, m, n are the direction cosines of PQ. If we suppose x, y, z and l, m, n to be functions of p, these three equations express the coordinates of any point on the ruled surface in terms of p and q. Using the method and notation of Art. 456, if r is the distance from P of the point where a ray is met by the shortest distance from a consecutive ray

$$-r=\frac{l_1x_1+m_1y_1+n_1z_1}{l_1^{\,2}+m_1^{\,2}+n_1^{\,2}}.$$

The point thus defined is called the *central point* of the generator, and the tangent plane thereat the *central plane*.

Since $d\xi = (x_1 + ql_1)dp + ldq$, with similar values for $d\eta$ and $d\zeta$, the direction cosines of the normal to the tangent plane at ξ, η, ζ are proportional to the coefficients of $d\xi$, $d\eta$, $d\zeta$ in the determinant

$$\begin{vmatrix} d\xi & x_1+ql_1 & l \\ d\eta & y_1+qm_1 & m \\ d\zeta & z_1+qn_1 & n \end{vmatrix}$$

and the corresponding ratios for the tangent plane at x, y, z are formed by replacing q by zero. Now if P be the central point $l_1x_1+m_1y_1+n_1z_1=0$, and if in addition the ray is parallel to the axis of z, $l=m=0$ and $n=1$, and if, thirdly, the director curve be chosen so that its tangent line at P coincides with the axis of y, we have $x_1=z_1=0$. If all these conditions are fulfilled we have also $m_1=0$, and the direction cosines of the perpendicular to the tangent plane at any point ξ, η, ζ on the ray through P are in the ratio $y_1 : -ql_1 : 0$. Thus *the tangent of the angle between the tangent plane at any point of the generator and the tangent plane at the central point has a constant ratio to the distance between the points.* The reciprocal of this constant ratio has been called the *parameter of distribution* of the ray. It becomes infinite for a developable surface.

The locus of the central points is the line of striction (Art. 462) and its parametric equation is, in general,

$$q+\frac{l_1x_1+m_1y_1+n_1z_1}{l_1^2+m_1^2+n_1^2}=0.$$

Ex. 1. In general the parameter of distribution is

$$\pm\frac{1}{l_1^2+m_1^2+n_1^2}\begin{vmatrix} x_1 & y_1 & z_1 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix}$$

Ex. 2. Investigate by means of torsion and curvature the central points and parameters of distribution for the ruled surface generated by the principal normals of a curve.

If dt represent the shortest distance between two consecutive generators, and if $d\sigma^2 = dl^2 + dm^2 + dn^2$, it may be shown easily that the parameter of distribution is $\dfrac{dt}{d\sigma}$. Let us now consider the ruled surfaces of a rectilinear congruence which

passes through a given ray. Using the method of Art. 457, we can show that the parameter of distribution for a ruled surface corresponding to a given value of $dp : dq$ is

$$\frac{1}{(eg-f^2)^{\frac{1}{2}}}\;\frac{(eb'-af)dp^2+\{ec-ag+f(b'-b)\}dpdq+(cf-bg)dq^2}{edp^2+2fdpdq+cdq^2}$$

and thus, following the method of Art. 456, we can show

$$\pi=\pi_1\cos^2\theta+\pi_2\sin^2\theta$$

where θ is the angle which the central plane of the ruled surface with parameter of distribution π makes with a fixed plane. π_1 and π_2 are the extreme values of the parameter of distribution. It is easy to see that for a normal congruence $\pi_1+\pi_2=0$, for a hyperbolic congruence (Art. 457*a*) $\pi_1\pi_2$ is negative and for an elliptic positive, while for an isotropic or circular congruence $\pi_1=\pi_2$.] *

463. Given any generator of a ruled surface, we can describe a hyperboloid of one sheet, which shall have this generator in common with the ruled surface, and which shall also have the same tangent plane with that surface at every point of their common generator. For it is evident from the construction of Art. 458 that the tangent plane at every point on a generator is fixed, when the two next consecutive generators are given, and consequently that if two ruled surfaces have three consecutive generators in common, they will touch all along the first of these generators. Now any three non-intersecting right lines determine a hyperboloid of one sheet (Art. 112); the hyperboloid then determined by any generator and the two next consecutive will touch the given surface as required.

In order to see the full bearing of the theorem here enunciated, let us suppose that the axis of z lies altogether in any surface of the n^{th} degree, then every term in its equation must contain either x or y; and that equation arranged according to the powers of x and y will be of the form

$$u_{n-1}x+v_{n-1}y+u_{n-2}x^2+v_{n-2}xy+w_{n-2}y^2+\&c.=0,$$

* See an interesting paper by Sannia, "Geometria differenziale delle congruenze rettilinee" (*Math. Ann.*, 68, 1910).

where u_{n-1}, v_{n-1} denote functions of z of the $(n-1)^{th}$ degree, &c. Then (see Art. 110) the tangent plane at any point on the axis will be $u'_{n-1}x + v'_{n-1}y = 0$, where u'_{n-1} denotes the result of substituting in u_{n-1} the coordinates of that point. Conversely, it follows that any plane $y = mx$ touches the surface in $n-1$ points, which are determined by the equation $u_{n-1} + mv_{n-1} = 0$. If however u_{n-1}, v_{n-1} have a common factor u_p, so that the terms of the first degree in x and y may be written $u_p(u_{n-p-1}x + v_{n-p-1}y) = 0$, then the equation of the tangent plane will be $u'_{n-p-1}x + v'_{n-p-1}y = 0$, and evidently in this case any plane $y = mx$ will touch the surface only in $n-p-1$ points. It is easy to see that the points on the axis for which $u_p = 0$ are double points on the surface. Now what is asserted in the theorem of this article is, that when the axis of z is not an isolated right line on a surface, but one of a system of right lines by which the surface is generated, then the form of the equation will be

$$u_{n-2}(ux + vy) + \&c. = 0,$$

so that the tangent plane at any point on the axis will be the same as that of the hyperboloid $ux + vy$, viz. $u'x + v'y = 0$. And any plane $y = mx$ will touch the surface in but one point. The factor u_{n-2} indicates that there are on each generator $n-2$ points which are double points on the surface.

[Ex. The tangent lines to the second system of asymptotic lines through the points on the given generator are generators of the hyperboloid.]

464. We can verify the theorem just stated, for an important class of ruled surfaces, viz., those of which any generator can be expressed, by two equations of the form

$$at^m + bt^{m-1} + ct^{m-2} + \&c. = 0,\ a't^n + b't^{n-1} + c't^{n-2} + \&c. = 0,$$

where a, a', b, b', &c. are linear functions of the coordinates, and t a variable parameter. Then the equation of the surface obtained by eliminating t between the equations of the generator (see *Higher Algebra*, Arts. 85, 86) may be written in the form of a determinant, of which when $m = n$ the first row and first column are identical, being (ab'), (ac'), (ad'), &c., or when $m > n$, the first row is as before and the first column consists

of n such constituents, a' and zeros. Now the line aa' is a generator, namely, that answering to $t=\infty$; and we have just proved that either a or a' will appear in every term, both of the first row and of the first column. Since, then, every term in the expanded determinant contains a factor from the first row and a factor from the first column, the expanded determinant will be a function of, at least, the second degree in a and a', except that part of it which is multiplied by (ab'), the term common to the first row and first column. But that part of the equation which is only of the first degree in a and a' determines the tangent at any point of aa'; the ruled surface is therefore touched along that generator by the hyperboloid $ab'-ba'=0$.

If a and b (or a' and b') represent the same plane, then the generator aa' intersects the next consecutive, and the plane a touches along its whole length. If we had $b=ka$, $b'=ka'$, the terms of the first degree in a and a' would vanish, and aa' would be a double line on the surface.

465. Returning to the theory of ruled surfaces in general, it is evident that any plane through a generator meets the surface in that generator and in a curve of the $(n-1)^{\text{th}}$ order meeting the generator in $n-1$ points. Each of these points being a double point in the curve of section is (Art. 264) in a certain sense a point of contact of the plane with the surface. But we have seen (Art. 463) that only one of them is properly a point of contact of the plane; the other $n-2$ are fixed points on the generator, not varying as the plane through it is changed. They are the points where this generator meets other non-consecutive generators, and are points of a double curve on the surface. Thus, then, *a skew ruled surface in general has a double curve which is met by every generator in $n-2$ points.* It may of course happen, that two or more of these $n-2$ points coincide, and the multiple curve on the surface may be of higher order than the second. In the case considered in the last article, it can be proved (see *Higher*

Algebra, Lesson XVIII.) that the multiple curve is of the order

$$\tfrac{1}{2}(m+n-1)(m+n-2),$$

and that the number of triple points thereon is

$$\tfrac{1}{6}(m+n-2)(m+n-3)(m+n-4).$$

A ruled surface having a double line will in general not have any cuspidal line unless the surface be a developable, and the section by any plane will therefore be a curve having double points but not cusps.

466. Consider now the cone whose vertex is any point, and which envelopes the surface. Since every plane through a generator touches the surface in some point, the tangent planes to the cone are the planes joining the series of generators to the vertex of the cone. The cone will, in general, not have any stationary tangent planes; for such a plane would arise when two consecutive generators lie in the same plane passing through the vertex of the cone. But it is only in special cases that a generator will be intersected by one consecutive; the number of planes through two consecutive generators is therefore finite; and hence, one will, in general, not pass through an assumed point. The class of the cone, being equal to the number of tangent planes which can be drawn through any line through the vertex, is equal to the number of generators which can meet that line, that is to say, to the degree of the surface (see end of Art. 124). We have proved now that the *class* of the cone is equal to the degree of a section of the surface; and that the former has no stationary tangent planes as the latter has no stationary or cuspidal points. The equations then which connect any three of the singularities of a curve prove that the number of double tangent planes to the cone must be equal to the number of double points of a section of the surface; or, in other words, that the number of planes containing two generators which can be drawn through an assumed point, is equal to the number of points of intersection of two generators which lie in an assumed plane.*

* These theorems are Cayley's. *Cambridge and Dublin Mathematical Journal*, vol. VII. p. 171.

467. We shall illustrate the preceding theory by an enumeration of some of the singularities of *the ruled surface generated by a line meeting three fixed directing curves*, the degrees of which are m_1, m_2, m_3.*

The degree of the surface generated is equal to the number of generators which meet an assumed right line; it is therefore equal to the number of intersections of the curve m_1 with the ruled surface having for directing curves the curves m_2, m_3 and the assumed line; that is to say, it is m_1 times the degree of the latter surface. The degree of this again is, in like manner, m_2 times the degree of the ruled surface whose directing curves are two right lines and the curve m_3, while by a repetition of the same argument, the degree of this last surface is $2m_3$. It follows that the degree of the ruled surface when the generators are curves m_1, m_2, m_3, is $2m_1m_2m_3$.

The three directing curves are multiple lines on the surface, whose orders of multiplicity are respectively m_2m_3, m_3m_1, m_1m_2. For through any point on the first curve pass m_2m_3 generators, the intersections, namely, of the cones having this point for a common vertex, and resting on the curves m_2, m_3.

468. The degree of the ruled surface, as calculated by the last article, will admit of reduction if any pair of the directing curves have points in common. Thus, if the curves m_2, m_3 have a point in common, it is evident that the cone whose vertex is this point, and base the curve m_1, will be included in the system, and that the degree of the ruled surface proper will be reduced by m_1, while the curve m_1 will be a multiple line whose order of multiplicity is only $m_2m_3 - 1$. And generally if the three pairs made out of the three directing curves have common respectively α, β, γ points, the degree of the ruled surface will be reduced by $m_1\alpha + m_2\beta + m_3\gamma$,† while the order

* I published a discussion of this surface, *Cambridge and Dublin Mathematical Journal*, vol. VIII. p. 45.

† My attention was called by Prof. Cayley to this reduction, which takes place when the directing curves have points in common.

of multiplicity of the directing curves will be reduced respectively by α, β, γ. Thus, if the directing lines be two right lines and a twisted cubic, the surface is in general of the sixth degree, but if each of the lines intersect the cubic, the degree is only the fourth. If each intersect it twice, the surface is a quadric. If one intersect it twice and the other once, the surface is a skew surface of the third degree on which the former line is a double line.

Again, let the directing curves be any three plane sections of a hyperboloid of one sheet. According to the general theory the surface ought to be of the sixteenth degree, and let us see how a reduction takes place. Each pair of directing curves have two points common; namely, the points in which the line of intersection of their planes meets the surface. And the complex surface of the sixteenth degree consists of six cones of the second degree, together with the original quadric reckoned twice. That it must be reckoned twice, appears from the fact that the four generators which can be drawn through any point on one of the directing curves are two lines belonging to the cones and *two* generators of the given hyperboloid.

In general, if we take as directing curves three plane sections of any ruled surface, the equation of the ruled surface generated will have, in addition to the cones and to the original surface, a factor denoting another ruled surface which passes through the given curves. For it will generally be possible to draw lines, meeting all three curves which are not generators of the original surface.

469. The degree of the ruled surface being $2m_1m_2m_3$, it follows, from Art. 465, that any generator is intersected by $2m_1m_2m_3 - 2$ other generators. But we have seen that at the points where it meets the directing curves, it meets $(m_2m_3 - 1) + (m_3m_1 - 1) + (m_1m_2 - 1)$ other generators. Consequently it must meet $2m_1m_2m_3 - (m_2m_3 + m_3m_1 + m_1m_2) + 1$ generators, in points not on the directing curves. We shall establish this result independently by seeking the number of generators

which can meet a given generator. By the last article, the degree of the ruled surface whose directing curves are the curves m_1, m_2, and the given generator, which is a line resting on both, is $2m_1m_2 - m_1 - m_2$. Multiplying this number by m_3, we get the number of points where this new ruled surface is met by the curve m_3. But amongst these will be reckoned $(m_1m_2 - 1)$ times the point where the given generator meets the curve m_3. Subtracting this number, then, there remain

$$2m_1m_2m_3 - m_2m_3 - m_1m_3 - m_1m_2 + 1$$

points of the curve m_3, through which can be drawn a line to meet the curves m_1, m_2, and the assumed generator. But this is in other words the thing to be proved.

470. We can examine in the same way the degree of the *surface generated by a line meeting a curve m_1 twice, and another curve m_2 once.* It is proved, as in Art. 467, that the degree is m_2 times the degree of the surface generated by a line meeting m_1 twice, and meeting any assumed right line. Now if h_1 be the number of apparent double points of the curve m_1, that is to say, the number of lines which can be drawn through an assumed point to meet that curve twice, it is evident that the assumed right line will on this ruled surface be a multiple line of the order h_1, and the section of the ruled surface by a plane through that line will be that line h_1 times, together with the $\frac{1}{2}m_1\ (m_1 - 1)$ lines joining any pair of the points where the plane cuts the curve m_1. The degree of this ruled surface will then be $h_1 + \frac{1}{2}m_1\ (m_1 - 1)$, and, as has been said, the degree will be m_2 times this number, if the second director be a curve m_2 instead of a right line.

The result of this article may be verified as follows: Consider a complex curve made up of two simple curves m_1, m_2; then a line which meets this system twice must either meet both the simple curves, or else must meet one of them twice. The number of apparent double points of the system is $h_1 + h_2 + m_1m_2$;* and the degree of the surface generated by

* Where I use h in these formulæ Prof. Cayley uses r, the rank of the system, substituting for h from the formula $r = m\ (m - 1) - 2h$. And when the system is a complex one, we have simply $R = r_1 + r_2$.

a line meeting a right line, and meeting the complex curve twice, is

$$\tfrac{1}{2}(m_1+m_2)(m_1+m_2-1)+h_1+h_2+m_1m_2$$
$$=\{\tfrac{1}{2}m_1(m_1-1)+h_1\}+\{\tfrac{1}{2}m_2(m_2-1)+h_2\}+2m_1m_2.$$

471. The degree of the *surface generated by a line which meets a curve three times* may be calculated as follows, when the curve is given as the intersection of two surfaces U, V: Let $x'y'z'w'$ be any point on the curve, $xyzw$ any point on a generator through $x'y'z'w'$; and let us, as in Art. 343, form the two equations $\delta U'+\frac{1}{2}\lambda\delta^2U'+\&c.=0$, $\delta V'+\frac{1}{2}\delta^2V'+\&c.=0$. Now if the generator meet the curve twice again, these equations must have two common roots. If then we form the conditions that the equations shall have two common roots, and between these and $U'=0$, $V'=0$, eliminate $x'y'z'w'$, we shall have the equation of the surface; or, rather that equation three times over, since each generator corresponds to three different points on the curve UV. But since U' and V' do not contain $xyzw$, the degree of the result of elimination will be the product of pq the order of U', V' by the *weight* of the other two equations (see *Higher Algebra*, Lesson XVIII.). If, then, we apply the formulæ given in that Lesson for finding the weight of the system of conditions that two equations shall have two common roots, putting $m=p-1$, $n=q-1$, $\lambda=0$, $\lambda'=p$, $\mu=0$, $\mu'=q$, the result is

$$\tfrac{1}{2}(pq-2)\{2pq-3(p+q)+4\},$$

and the degree of the required surface is this number multiplied by $\frac{1}{3}pq$. But the intersection of U, V is a curve (see Art. 343), for which $m=pq$, $2h=pq(p-1)(q-1)$, whence $pq(p+q)=m^2+m-2h$. Substituting these values, the degree of the surface expressed in terms of m and h is

$$\tfrac{1}{6}(m-2)(6h+m-m^2), \text{ or } (m-2)h-\tfrac{1}{6}m(m-1)(m-2),$$

a number which may be verified, as in the last article.

472. The ruled surfaces considered in the preceding articles have all a certain number of double generators. Thus, if a line meets the curve m_1 twice, and also the curves m_2 and m_3, it belongs doubly to the system of lines which

meet the curves m_1, m_2, m_3 and is a double generator on the corresponding surface. But the number of such lines is evidently equal to the number of intersections of the curve m_3, with the surface generated by the lines which meet m_1 twice, and also m_2, that is to say, is $m_2m_3\{\frac{1}{2}m_1(m_1-1)+h_1\}$; the total number of double generators is therefore

$$\tfrac{1}{2}m_1m_2m_3(m_1+m_2+m_3-3)+h_1m_2m_3+h_2m_3m_1+h_3m_1m_2.$$

In like manner the lines which meet m_1 three times, and also m_2 belong triply to the system of lines which meet m_1 twice, and also m_2; and the number of such triple generators is seen by the last article to be $m_2(m_1-2)h_1-\frac{1}{6}m_1m_2(m_1-1)\ (m_1-2)$. The surface has also double generators whose number we shall determine presently, being the lines which meet both m_1 and m_2 twice.

Lastly, the lines which meet a curve four times are multiple lines of the fourth order of multiplicity on the surface generated by the lines which meet the curve three times. We can determine the number of such lines when the curve is given as the intersection of two surfaces, but will first establish a principle which admits of many applications.

473. Let the equations of three surfaces U, V, W contain $xyzw$ in the degrees respectively λ, λ', λ'', and $x'y'z'w'$ in degrees μ, μ', μ'', and let the $\lambda\lambda'\lambda''$ points of intersection of these surfaces all coincide with $x'y'z'w'$; then it is required to find the degree of the further condition which must be fulfilled in order that they may have a line in common. When this is the case, any arbitrary plane $\alpha x+\beta y+\gamma z+\delta w$ must be certain to have a point in common with the three surfaces (namely, the point where it is met by the common line), and therefore the result of elimination between U, V, W and the arbitrary plane must vanish. This result is of the degree $\lambda\lambda'\lambda''$ in $\alpha\beta\gamma\delta$, and $\mu\lambda'\lambda''+\mu'\lambda''\lambda+\mu''\lambda\lambda'$ in $x'y'z'w'$. The first of these numbers (see *Higher Algebra*, Lesson XVIII.) we call the *order*, and the second the *weight* of the resultant. Now, since the resultant is obtained by multiplying together the results of substituting in $\alpha x+\beta y+\gamma z+\delta w$, the coordinates

of each of the points of intersection of U, V, W, this resultant must be of the form $\Pi\,(ax'+\beta y'+\gamma z'+\delta w')^{\lambda\lambda'\lambda''}$. The condition $ax'+\beta y'+\gamma z'+\delta w'=0$, merely indicates that the arbitrary plane passes through $x'y'z'w'$, in which case it passes through a point common to the three surfaces, whether they have a common line or not. The condition, therefore, that they shall have a common line is $\Pi=0$; and this must be of the degree

$$\mu\lambda'\lambda''+\mu'\lambda''\lambda+\mu''\lambda\lambda'-\lambda\lambda'\lambda'';$$

that is to say, *the degree of the condition is got by subtracting the order from the weight of the equations* U, V, W.

474. Now let $x'y'z'w'$ be any point on the curve of intersection of two surfaces U, V, $xyzw$ any other point; and, as in Art. 471, let us form the equations $\delta U+\frac{1}{2}\lambda\delta^2 U+\&c.=0$, $\delta V+\frac{1}{2}\lambda\delta^2 V+\&c.=0$. If $x'y'x'w'$ be a point through which a line can be drawn to meet the curve in four points, and $xyzw$ any point whatever on that line, these two equations in λ will have three roots common. And, therefore, if we form the three conditions that the equations should have three roots common, these conditions considered as functions of $xyzw$, denote surfaces having common the line which meets the curve in four points. But if $x'y'z'w'$ had not been such a point, it would not have been possible to find any point $xyzw$ distinct from $x'y'z'w'$, for which the three conditions would be fulfilled; and, therefore, in general the conditions denote surfaces having no point common but $x'y'z'w'$. The degree, then, of the condition which $x'y'z'w'$ must fulfil, if it be a point through which a line can be drawn to meet the curve in four points, is, by the last article, the difference between the weight and the order of the system of conditions, that the equations should have three common roots. But (see *Higher Algebra*, Lesson XVIII.) the weight of this system of conditions is found, by making $m=p-1$, $n=q-1$, $\lambda=p$, $\mu=q$, $\lambda'=\mu'=0$, to be

$$\tfrac{1}{6}\{3p^3q^3-9p^2q^2(p+q)+2p^2q^2+5pq(p+q)^2$$
$$+15pq\,(p+q)-13pq-66\,(p+q)+108\};$$

while the order of the same system is

$$\tfrac{1}{6}\{p^3q^3 - 3p^2q^2(p+q) + 2p^2q^2 + 2pq\,(p+q)^2 - 3pq\,(p+q) + 13pq - 36\}.$$

The degree, then, of the condition $\Pi = 0$ to be fulfilled by $x'y'z'w'$, being the difference of these numbers, is

$$\tfrac{1}{6}\{2p^3q^3 - 6p^2q^2(p+q) + 3pq\,(p+q)^2 + 18pq\,(p+q) - 26pq - 66\,(p+q) + 144\}.$$

The intersection of the surface Π with the given curve determines the points through which can be drawn lines to meet in four points; and the number of such lines is therefore $\frac{1}{4}$ of the number just found multiplied by pq. As before, putting $pq = m$, $pq\,(p+q) = m^2 + m - 2h$, the number of lines meeting in four points is found to be

$$\tfrac{1}{24}\{-m^4 + 18m^3 - 71m^2 + 78m - 48mh + 132h + 12h^2\}.*$$

From this number can be derived the number of lines which meet both of two curves twice. For, substitute in the formula just written $m_1 + m_2$ for m, and $h_1 + h_2 + m_1m_2$ for h, and we have the number of lines which meet the complex curve four times. But from this take away the number of lines which meet each four times, and the number given (Art. 472) of those which meet one three times and the other once; and the remainder is the number of lines which meet both curves twice, viz.

$$h_1h_2 + \tfrac{1}{4}m_1m_2\,(m_1 - 1)\,(m_2 - 1).$$

475. Besides the multiple generators, the ruled surfaces we have been considering have also nodal curves, being the locus of points of intersection of two different generators. I do not know any direct method of obtaining the order of these nodal curves; but Cayley has succeeded in arriving at a solution of the problem by the following method. Let m be one of the curves used in generating one of the surfaces

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* It *may* happen, as Cayley has remarked, that the surface Π may altogether contain the given curve, in which case an infinity of lines can be drawn to meet in four points. Thus the curve of intersection of a ruled surface by a surface of the p_{th} order is evidently such that every generator of the ruled surface meets the curve in p points.

we have been considering, M the degree of that surface, $\phi(m)$ the degree of the aggregate of all the double lines on that surface; then if we suppose m to be a complex curve made up of two simple curves m_1 and m_2, the surface will consist of two surfaces M_1, M_2 having as a double line the intersection of M_1 and M_2, in addition to the double lines on each surface. Thus, then, $\phi(m)$ must be such as to satisfy the condition

$$\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2) + M_1 M_2.$$

Using, then, the value already found for M_1 in terms of m_1, solving this functional equation, and determining the constants involved in it by the help of particular cases in which the problem can be solved directly, Cayley arrives at the conclusion, that the order of the nodal curve, distinct from the multiple generators, is in the case of the surface generated by a line meeting three curves m_1, m_2, m_3,

$$\tfrac{1}{2}m_1m_2m_3\{4m_1m_2m_3 - (m_2m_3 + m_3m_1 + m_1m_2) \\ - 2(m_1 + m_2 + m_3) + 5\},$$

in the case of the surface generated by a line meeting m_1 twice and m_2 once, is

$$m_2\{\tfrac{1}{2}h_1(m_1 - 2)(m_1 - 3) + \tfrac{1}{8}m_1(m_1 - 1)(m_1 - 2)(m_1 - 3)\} \\ + m_2(m_2 - 1)\{\tfrac{1}{2}h_1^2 + \tfrac{1}{2}h_1(m_1^2 - m_1 - 1) + \tfrac{1}{8}m_1(m_1 - 1)(m_1^2 - 5m_1 + 10)\},$$

and in the case of the surface generated by a line meeting m_1 three times, is

$$\tfrac{1}{2}h_1^2 m_1(m_1 - 5) - \tfrac{1}{6}h_1(m_1^4 - 5m_1^3 + 5m_1^2 - 49m_1 + 120) \\ + \tfrac{1}{72}(m_1^6 - 6m_1^5 + 31m_1^4 - 270m_1^3 + 868m_1^2 - 408m_1).$$

CHAPTER XIII (*b*).

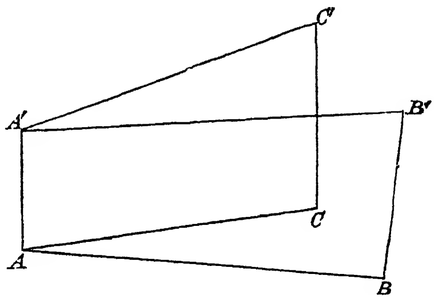
TRIPLY ORTHOGONAL SYSTEMS OF SURFACES.*—NORMAL CONGRUENCES OF CURVES.

476. We have already given proofs of Dupin's theorem regarding orthogonal surfaces in Art. 304; as this theorem has led to investigations on systems of orthogonal surfaces, we proceed to present the proof under a different and somewhat more geometrical form as follows. Imagine a given surface, and on each normal measure off from the surface an infinitesimal distance l (varying at pleasure from point to point of the surface, or say an arbitrary function of the position of the point on the surface): the extremities of these distances form a new surface, which may be called the consecutive surface; and to each point of the given surface corresponds a point on the consecutive surface, viz. the point on the normal at the distance l; hence, to any curve or series of curves on the given surface corresponds a curve or series of curves on the consecutive surface. Suppose that we have on the given surface two series of curves cutting at right angles, then we have on the consecutive surface the corresponding two series of curves, but these will not in general intersect at right angles.

Take A a point on the given surface; AB, AC, elements of the two curves through A; AA', BB', CC' the infinitesimal distances on the three normals; then we have on the consecutive surface the point A', and the elements $A'B'$, $A'C'$ of the two corresponding curves; the angles at A are by hypo-

* [By a triply orthogonal system of surfaces is meant three one-parameter families of surfaces, such that each surface cuts orthogonally all those of different family. Any one of the families is described as a Lamé family.]

thesis each of them a right angle; the angle $B'A'C'$ is not in general a right angle, and it may be shown that the condition of its being so, is that the normals BB', AA' shall intersect, or that the normals CC', AA' shall intersect, for it can be shown that if one pair intersect, the other pair also intersect.



But the normals intersecting, AB, AC, will be elements of the lines of curvature, and the two series of curves on the given surface will be the lines of curvature of this surface.

477. Take x, y, z for the coordinates of the point A; α, β, γ for the direction-cosines of AA'; $\alpha_1, \beta_1, \gamma_1$ for those of AB, and $\alpha_2, \beta_2, \gamma_2$ for those of AC. Write also

$$\begin{aligned}\delta &= \alpha\, d_x + \beta\, d_y + \gamma\, d_z,\\ \delta_1 &= \alpha_1 d_x + \beta_1 d_y + \gamma_1 d_z,\\ \delta_2 &= \alpha_2 d_x + \beta_2 d_y + \gamma_2 d_z.\end{aligned}$$

Then it will be shown that the condition for the intersection of the normals AA', BB', is

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$

and that these are equivalent to each other, and to the condition for the angle $B'A'C'$ being a right angle.

Taking l, l_1, l_2 for the lengths AA', AB, AC, the coordinates of A', B, C measured from the point A, are respectively

$$(l\alpha,\ l\beta,\ l\gamma),\ (l_1\alpha_1,\ l_1\beta_1,\ l_1\gamma_1),\ (l_2\alpha_2,\ l_2\beta_2,\ l_2\gamma_2).$$

The equations of the normal at A may be written

$$X = x + \theta\alpha,\ Y = y + \theta\beta,\ Z = z + \theta\gamma,$$

where X, Y, Z are current coordinates, and θ is a variable parameter. Hence for the normal at B passing from the

coordinates x, y, z to $x+l_1\alpha_1$, $y+l_1\beta_1$, $z+l_1\gamma_1$, the equations are

$$\begin{aligned} X &= x+\theta\alpha + l_1\alpha_1 + l_1\delta_1(\theta\alpha),\\ Y &= y+\theta\beta + l_1\beta_1 + l_1\delta_1(\theta\beta),\\ Z &= z+\theta\gamma + l_1\gamma_1 + l_1\delta_1(\theta\gamma),\end{aligned}$$

and if the two normals intersect in the point (X, Y, Z), then

$$\begin{aligned} \alpha_1+\alpha\delta_1\theta+\theta\delta_1\alpha &= 0,\\ \beta_1+\beta\delta_1\theta+\theta\delta_1\beta &= 0,\\ \gamma_1+\gamma\delta_1\theta+\theta\delta_1\gamma &= 0.\end{aligned}$$

Eliminating θ and $\delta_1\theta$, the condition is

$$\begin{vmatrix} \alpha_1, & \alpha, & \delta_1\alpha \\ \beta_1, & \beta, & \delta_1\beta \\ \gamma_1, & \gamma, & \delta_1\gamma \end{vmatrix} = 0;$$

or since $\quad \alpha_2, \beta_2, \gamma_2 = \beta\gamma_1-\beta_1\gamma,\ \gamma\alpha_1-\gamma_1\alpha,\ \alpha\beta_1-\alpha_1\beta,$

this is
$$\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma=0.$$

Similarly the condition for the intersection of the normals AA', CC' is

$$\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma=0.$$

We have next to show that

$$\alpha_2\delta_1\alpha+\beta_2\delta_1\beta+\gamma_2\delta_1\gamma=\alpha_1\delta_2\alpha+\beta_1\delta_2\beta+\gamma_1\delta_2\gamma.$$

In fact, this equation is

$$(\alpha_2\delta_1-\alpha_1\delta_2)\,\alpha+(\beta_2\delta_1-\beta_1\delta_2)\,\beta+(\gamma_2\delta_1-\gamma_1\delta_2)\,\gamma=0,$$

which we proceed to verify.

In the first term the symbol $\alpha_2\delta_1-\alpha_1\delta_2$ is

$$\alpha_2\,(\alpha_1d_x+\beta_1d_y+\gamma_1d_z)-\alpha_1\,(\alpha_2d_x+\beta_2d_y+\gamma_2d_z),$$

this is
$$(\alpha_2\beta_1-\alpha_1\beta_2)\,d_y+(\gamma_1\alpha_2-\gamma_2\alpha_1)\,d_z;$$

or, what is the same thing, it is

$$\beta d_z-\gamma d_y,$$

and the equation to be verified is

$$(\beta d_z-\gamma d_y)\,\alpha+(\gamma d_x-\alpha d_z)\,\beta+(\alpha d_y-\beta d_x)\,\gamma=0.$$

Writing
$$\alpha, \beta, \gamma=\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R},$$

where if $l=f(x, y, z)$ is the equation of the surface, X, Y, Z are the derived functions $\dfrac{df}{dx}, \dfrac{df}{dy}, \dfrac{df}{dz}$, and $R=\sqrt{(X^2+Y^2+Z^2)}$, the function on the left-hand consists of two parts; the first is

$$\frac{1}{R}\{(\beta d_z - \gamma d_y)\,X + (\gamma d_x - \alpha d_z)\,Y + (\alpha d_y - \beta d_x)\,Z\},$$

that is $\frac{1}{R}\{\alpha\,(d_y Z - d_z Y) + \beta\,(d_z X - d_x Z) + \gamma\,(d_x Y - d_y X)\},$

which vanishes; and the second is

$$-\frac{1}{R}\{\alpha\,(\beta d_z - \gamma d_y) + \beta\,(\gamma d_x - \alpha d_z) + \gamma\,(\alpha d_y - \beta d_x)\}\,R,$$

which also vanishes; that is, we have identically

$$\alpha_2\delta_1\alpha + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = \alpha_1\delta_2\alpha + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma,$$

and the vanishing of the one function implies the vanishing of the other.

Proceeding now to the condition that the angle $B'A'C'$ shall be a right angle, the coordinates of B' are what those of A' become on substituting in them $x + l_1\alpha_1$, $y + l_1\beta_1$, $z + l_1\gamma_1$ in place of x, y, z; that is, these coordinates are

$$x + l\alpha + l_1\alpha_1 + l_1\delta_1(l\alpha),\ \&c.,$$

or, what is the same thing, measuring them from A' as origin, the coordinates of B' are

$$\begin{aligned} &l_1\,(\alpha_1 + l\delta_1\alpha + \alpha\delta_1 l),\\ &l_1\,(\beta_1 + l\delta_1\beta + \beta\delta_1 l),\\ &l_1\,(\gamma_1 + l\delta_1\gamma + \gamma\delta_1 l),\end{aligned}$$

and similarly those of C' measured from the same origin A are

$$\begin{aligned} &l_2\,(\alpha_2 + l\delta_2\alpha + \alpha\delta_2 l),\\ &l_2\,(\beta_2 + l\delta_2\beta + \beta\delta_2 l),\\ &l_2\,(\gamma_2 + l\delta_2\gamma + \gamma\delta_2 l).\end{aligned}$$

Hence the condition for the angle to be right is

$$\begin{aligned} &(\alpha_1 + l\delta_1\alpha + \alpha\delta_1 l)\,(\alpha_2 + l\delta_2\alpha + \alpha\delta_2 l)\\ &+ (\beta_1 + l\delta_1\beta + \beta\delta_1 l)\,(\beta_2 + l\delta_2\beta + \beta\delta_2 l)\\ &+ (\gamma_1 + l\delta_1\gamma + \gamma\delta_1 l)\,(\gamma_2 + l\delta_2\gamma + \gamma\delta_2 l) = 0.\end{aligned}$$

Here the terms independent of l, $\delta_1 l$, $\delta_2 l$ vanish; and writing down only the terms which are of the first order in these quantities, the condition is

$$\begin{aligned} &\alpha_1\,(l\delta_2\alpha + \alpha\delta_2 l) + \alpha_2\,(l\delta_1\alpha + \alpha\delta_1 l)\\ &+ \beta_1\,(l\delta_2\beta + \beta\delta_2 l) + \beta_2\,(l\delta_1\beta + \beta\delta_1 l)\\ &+ \gamma_1\,(l\delta_2\gamma + \gamma\delta_2 l) + \gamma_2\,(l\delta_1\gamma + \gamma\delta_1 l) = 0,\end{aligned}$$

where the terms in $\delta_1 l$, $\delta_2 l$ vanish; the remaining terms divide by l, and throwing out this factor, the condition is

$$(a_1\delta_2 a + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma) + (a_2\delta_1 a + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma) = 0.$$

By what precedes, this may be written under either of the forms

$$a_1\delta_2 a + \beta_1\delta_2\beta + \gamma_1\delta_2\gamma = 0,$$
$$a_2\delta_1 a + \beta_2\delta_1\beta + \gamma_2\delta_1\gamma = 0,$$

and the theorem is thus proved.

Now in any system of triply orthogonal surfaces taking for the given surface of the foregoing demonstration any surface of one family, we have not only on the given surface, but also on the consecutive surface of the family, two series of curves cutting at right angles; and the demonstrated property is that the two series of curves on the given surface (that is on any surface of the family) are the lines of curvature of the surface. And the same being of course the case as to the surfaces of the other two families respectively, we have Dupin's theorem.

478. In regard to the foregoing proof, it is important to remark that there is nothing to show, and it is not in fact in general the case, that $A'B'$, $A'C'$ are elements of the lines of curvature on the consecutive surface. The consecutive surface (as constructed with an arbitrarily varying value of l) is in fact *any* surface everywhere indefinitely near to the given surface; and since by hypothesis AA' and BB' intersect and also AA', CC' intersect, then AB and $A'B'$ intersect, and also AC and $A'C'$; the theorem, if it were true, would be, that taking on the given surface any point A, and drawing the normal to meet the consecutive surface in A', then the tangents AB, AC of the lines of curvature at A meet respectively the tangents $A'B'$, $A'C'$ of the lines of curvature through A'; and it is obvious that this is not in general the case; that it shall be so, implies a restriction on the arbitrary value of the function l.

Cayley has shown that when the position of the point A on the given surface is determined by the parameters p, q,

which are such that the equations of the curves of curvature are $p = \text{const.}$, $q = \text{const.}$ respectively, then the condition is that l shall satisfy the same partial differential equation as is satisfied by the coordinates x, y, z considered as functions of p, q, viz. the equation (Art. 384)

$$\frac{d^2u}{dp\,dq} - \frac{1}{2}\frac{1}{E}\frac{dE}{dq}\frac{du}{dp} - \frac{1}{2}\frac{1}{G}\frac{dG}{dp}\frac{du}{dq} = 0.$$

The above conclusion may be differently stated: taking $r = f(x, y, z)$ a perfectly arbitrary function of (x, y, z), the family of surfaces $r = f(x, y, z)$, does not belong to a system of orthogonal surfaces; in order that it may do so the foregoing property must hold good; viz. it is necessary that taking a point A on the surface r, and passing along the normal to the point A' on the consecutive surface $r + dr$, the tangents to the lines of curvature at A shall respectively meet the tangents to the lines of curvature at A'. And this implies that r, considered as a function of x, y, z, satisfies a certain partial differential equation of the third order, Cayley's investigation of which will be given presently.*

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* The remark that r is not a perfectly arbitrary function of (x, y, z) was first made by Bouquet, *Liouv.* t. XI. p. 446 (1846), and he also showed that in the particular case where r is of the form $r = f(x) + \phi(y) + \psi(z)$, the necessary condition was that r should satisfy a certain partial differential equation of the third order; this equation was found by him, and in a different manner by Serret, *Liouv.* t. XII. p. 241 (1847). That the same is the case generally was shown by Bonnet (*Comptes rendus*, LIV 556, 1862), and a mode of obtaining this equation is indicated by Darboux, *Ann. de l'école normale*, t. III. p. 110 (1866). His form of the theorem is that in the surface $r = f(x, y, z)$ if α, β, γ are the direction-cosines of a line of curvature at a given point of the surface, then the function must be such that the differential equation $\alpha dx + \beta dy + \gamma dz = 0$ shall be integrable by a factor. The condition as given in the text is in the form given by Levy, *Jour. de l'école polyt.*, XLIII. (1870); he does not obtain the partial differential equation though he finds what it becomes on writing therein $\frac{dr}{dx} = 0$, $\frac{dr}{dy} = 0$; the actual equation (which of course includes as well this result, as the particular case obtained by Bouquet and Serret) was obtained by Cayley, *Comptes rendus*, t. LXXV. (1872); but in a form which (as he afterwards discovered) was affected with an extraneous factor. [For a fuller historical account see Lévy, *op. cit.*, p. 127.]

479. Dupin's theorem, and the notion of orthogonal surfaces are the foundation of Lamé's theory of curvilinear coordinates.* Representing the three families of orthogonal surfaces by $p=\phi(x, y, z)$, $q=\psi(x, y, z)$, $r=f(x, y, z)$, then conversely x, y, z are functions of p, q, r which are said to be the curvilinear coordinates of the point. It will be observed that regarding one of the coordinates, say r, as an absolute constant, then p, q are parameters determining the position of the point on the surface $r=f(x, y, z)$, such as are used in Gauss' theory of the curvature of surfaces; and by Dupin's theorem it appears that on this surface the equations of the lines of curvature are $p=$ const. $q=$ const. respectively; whence also (Art. 384) x, y, z each satisfy the differential equation

$$\frac{d^2u}{dp\,dq}-\frac{1}{2}\frac{1}{E}\frac{dE}{dq}\frac{du}{dp}-\frac{1}{2}\frac{1}{G}\frac{dG}{dp}\frac{du}{dq}=0,$$

(and the like equations with q, r and r, p in place of p, q respectively), a result obtained by Lamé, but without the geometrical interpretation.

Conversely we may derive another proof of Dupin's theorem from these considerations; taking x, y, z as given functions of p, q, r, and writing

$$\frac{dx}{dp}\frac{dx}{dq}+\frac{dy}{dp}\frac{dy}{dq}+\frac{dz}{dp}\frac{dz}{dq}=[p\,.\,q],$$

$$\frac{dx}{dp}\frac{d^2x}{dq\,dr}+\frac{dy}{dp}\frac{d^2y}{dq\,dr}+\frac{dz}{dp}\frac{d^2z}{dq\,dr}=[p\,.\,qr],\ \&c.,$$

the conditions for the intersections at right angles may be written

$$[q\,.\,r]=0,\ [r\,.\,p]=0,\ [p\,.\,q]=0,$$

and the first two equations give

$$\frac{dx}{dr}:\frac{dy}{dr}:\frac{dz}{dr}=\frac{dy}{dp}\frac{dz}{dq}-\frac{dz}{dp}\frac{dy}{dq}:\frac{dz}{dp}\frac{dx}{dq}-\frac{dx}{dp}\frac{dz}{dq}:\frac{dx}{dp}\frac{dy}{dq}-\frac{dy}{dp}\frac{dx}{dq}.$$

Moreover, by differentiating the three equations with respect to p, q, r respectively, we find

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* Lamé, *Comptes rendus*, t. VI. (1838), and *Liouv.* t. V. (1840), and various later Memoirs; also *Leçons sur les coordonnées curvilignes*, Paris, 1859.

$[rp\,.\,q]+[pq\,.\,r]=0,\ \ [pq\,.\,r]+[qr\,.\,p]=0,\ \ [qr\,.\,p]+[rp\,.\,q]=0,$ that is $[qr\,.\,p]=0$, $[rp\,.\,q]=0$, $[pq\,.\,r]=0$. The last of these equations, substituting in it for $\frac{dx}{dr}$, $\frac{dy}{dr}$, $\frac{dz}{dr}$ the foregoing values, becomes

$$\begin{vmatrix} \dfrac{dx}{dp}, & \dfrac{dy}{dp}, & \dfrac{dz}{dp} \\ \dfrac{dx}{dq}, & \dfrac{dy}{dq}, & \dfrac{dz}{dq} \\ \dfrac{d^2x}{dpdq}, & \dfrac{d^2y}{dpdq}, & \dfrac{d^2z}{dpdq} \end{vmatrix} = 0,$$

and the equation $[p\,.\,q]=0$ is

$$\frac{dx}{dp}\frac{dx}{dq}+\frac{dy}{dp}\frac{dy}{dq}+\frac{dz}{dp}\frac{dz}{dq}=0.$$

These equations are therefore satisfied by the values of x, y, z in terms of p, q, r; and regarding in them r as a given constant but p, q as variable parameters, the values in question represent a determinate surface of the family $r=f(x, y, z)$; and it thus appears that this surface is met in its lines of curvature by the surfaces of the other two families.

480. We proceed now to the investigation of Cayley's differential equation already referred to. Let P be a point on a surface belonging to a triply orthogonal system, PN the normal, PT_1, PT_2 the principal tangents or directions of curvature, then, by Dupin's theorem, the tangent planes to the two orthotomic surfaces are NPT_1, NPT_2. Take now a surface passing through a consecutive point P' on the normal, and if the surface be a consecutive one of the same orthogonal family, the planes NPT_1, NPT_2 must also meet *its* tangent plane at P' in the two principal tangents $P'T_1'$, $P'T_2'$. This is the condition which we are about to express analytically.

Take $r-f(x, y, z)=0$ for the equation of the family of the orthogonal system, the given surface being that corresponding to a given value of the parameter r; and let the differential coefficients of f (or what is the same thing, of r considered as a function of x, y, z) be L, M, N of the first

order, and a, b, c, f, g, h of the second order; and then the point P being taken as origin, the equation of the tangent plane at that point is $Lx + My + Nz = 0$, which we shall call for shortness $T = 0$; while the inflexional tangents are determined as the intersections of T with the cone

$$(a, b, c, f, g, h)(x, y, z)^2 = 0,$$

which we shall call $U = 0$. The two principal tangents are determined as being harmonic conjugates with the inflexional tangents, and also as being at right angles, that is to say, harmonic conjugates with the intersection of the plane T with $x^2 + y^2 + z^2 = 0$, or $V = 0$. Suppose now that we had formed the equation of the pair of planes through the normal, and through the inflexional tangents at P', and that this was

$$(a'', b'', c'', f'', g'', h'')(x, y, z)^2 = 0, \text{ or } W = 0,$$

then the planes NPT_1, NPT_2 must be harmonic conjugates with these also, so that the resulting condition is obtained by expressing that the three cones U, V, W intersect the plane T in three pairs of lines which form a system in involution.

Now we have here evidently to deal with the same analytical problem as that considered, *Conics*, Art. 388*c*, viz. to find the condition that three conics shall be met by a line in three pairs of points forming an involution. The general condition there given is applied to the present case by writing $a' = b' = c' = 1, f' = g' = h' = 0$, and in the determinant form is

$$\begin{vmatrix} a'', & b'', & c'', & 2f'', & 2g'', & 2h'' \\ a, & b, & c, & 2f, & 2g, & 2h \\ 1, & 1, & 1, & 0, & 0, & 0 \\ L, & 0, & 0, & 0, & N, & M \\ 0, & M, & 0, & N, & 0, & L \\ 0, & 0, & N, & M, & L, & 0 \end{vmatrix} = 0.$$

We see then that the form of the required condition is

$$\mathfrak{A}a'' + \mathfrak{B}b'' + \mathfrak{C}c'' + 2\mathfrak{F}f'' + 2\mathfrak{G}g'' + 2\mathfrak{H}h'' = 0,*$$

* Cayley has also shown, that if from any surface a new surface be derived by taking on each normal an infinitesimal distance $= \rho$, where ρ is a given function of x, y, z, the condition that the new surface shall belong to the same orthogonal system is

where $\mathfrak{A}$, $\mathfrak{B}$, &c., are the minors of the above written determinant, and it still remains to determine a'', b'', &c.

481. It may be observed, in the first instance, that the equation of the pair of planes passing through the normal, and the first pair of inflexional tangents is got by eliminating θ between $T+\theta T'=0$, $U+2\Pi\theta+\theta^2U'=0$, where T' is $L^2+M^2+N^2$, Π is

$$x(aL+hM+gN)+y(hL+bM+fN)+z(gL+fM+cN)$$

and U' is $aL^2+bM^2+cN^2+2fMN+2gNL+2hLM$.
The equation of the pair of planes is therefore

$$T'^2U-2\Pi TT'+T^2U'=0.$$

Now the consecutive point P' is a point on the normal whose coordinates may be taken as λL, λM, λN, λ being an infinitesimal whose square may be neglected, and the corresponding differential coefficients for the new point are $L+\lambda\delta L$, $M+\lambda\delta M$, $N+\lambda\delta N$, $a+\lambda\delta a$, &c., where δ denotes the operator

$$L\frac{d}{dx}+M\frac{d}{dy}+N\frac{d}{dz}.$$

Hence the equation of the tangent plane at P' referred to that point as origin, is $L'x+M'y+N'z=0$, or $T+\lambda\delta T=0$, where δT means $x\delta L+y\delta M+z\delta N$, and it is to be observed that δT is the same as what we have just called Π. And the equation of the cone which determines the inflexional tangents is $U+\lambda\delta U=0$. The equations of this plane and cone referred to the original axes are $T+\lambda(\delta T-T')=0$, $U+\lambda(\delta U-2\Pi)=0$, but it will be seen presently that the terms added on account of a change of origin do not affect the result. In order to form the equation of the pair of planes through the normal and through these inflexional tangents, we have to eliminate θ between

$$T+\lambda(\Pi-T')+\theta(T'+\&c.)=0,$$
$$U+\lambda(\delta U-2\Pi)+2\theta(\Pi+\&c.)+\theta^2(U'+\&c.)=0.$$

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^2\rho=0,$$

and that this condition is equivalent to that given in the text.

Now since we are about to express the condition that the resulting equation shall denote a surface intersecting T in a pair of lines belonging to an involution, to which the intersection of U by T also belongs, we need not attend to any terms in the result which contain either T or U; nor need we attend to any terms which contain more than the first power of λ. The terms then, of which alone we need take account, are

$$-2\Pi T'(\Pi - T') + T'^2(\delta U - \Pi) = 0,$$

or dividing by T', $\qquad T'\delta U - 2\Pi^2 = 0.$

We have thus $a'' = (L^2 + M^2 + N^2)\,\delta a - 2\,(\delta L)^2$, &c., and the required condition is

$$(L^2 + M^2 + N^2)\,(\mathfrak{A}\delta a + \mathfrak{B}\delta b + \mathfrak{C}\delta c + 2\mathfrak{F}\delta f + 2\mathfrak{G}\delta g + 2\mathfrak{H}\delta h)$$
$$= 2\,(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H})(\delta L,\ \delta M,\ \delta N)^2.$$

Cayley has shown that the condition originally obtained by him in a form equivalent to that just written, contains an irrelevant factor, the right-hand side of the equation being divisible by $L^2 + M^2 + N^2$. This we proceed to show.

482. We may in the first place remark, that since the united points or foci of an involution given by the two equations $u = (a,\ h,\ b)(x,\ y)^2$, $v = (a',\ h',\ b')(x,\ y)^2$, are determined by the equation $\begin{vmatrix} u_1, & u_2 \\ v_1, & v_2 \end{vmatrix} = 0$, *Conics*, Art. 342; if u and v be given as functions of $x,\ y,\ z$, where $Lx + My + Nz = 0$, and therefore $u_1 = \dfrac{du}{dx} - \dfrac{L}{N}\dfrac{du}{dz}$, &c., we find immediately that the foci of the involution are given by the equation

$$\begin{vmatrix} u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \\ L, & M, & N \end{vmatrix} = 0.$$

Thus then, or as in Art. 297, the two principal tangents are determined as the intersections of the tangent plane with the cone

$$\begin{vmatrix} ax + hy + gz, & hx + by + fz, & gx + fy + cz \\ x\quad, & y\quad, & z \\ L\quad, & M\quad, & N \end{vmatrix} = 0.$$

We shall write this equation

$$\tfrac{1}{2}\,(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(x, y, z)^2 = 0,$$

that is to say,

$$\mathrm{a} = 2\,(Mg - Nh),\ \mathrm{b} = 2\,(Nh - Lf),\ \mathrm{c} = 2\,(Lf - Mg),$$
$$\mathrm{f} = L\,(b - c) + Ng - Mh,\ \mathrm{g} = M\,(c - a) + Lh - Nf,$$
$$\mathrm{h} = N\,(a - b) + Mf - Lg.$$

It is useful to remark that the conic derived from two others, according to the rule just stated, viz. which is the Jacobian of two conics and of an arbitrary line, is connected with each of the two conics by the invariant relation $\Theta = 0$; that is to say, the two relations are

$$A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h} = 0,$$

where A, B, &c., are the reciprocal coefficients $bc - f^2$, &c.; and $A'\mathrm{a} +$ &c. $= 0$, which, in the particular case under consideration, reduces to $\mathrm{a} + \mathrm{b} + \mathrm{c} = 0$, which is manifestly true.

Again, referring to the condition, Art. 480, that three conics U, V, W should be met by a line in three pairs of points forming an involution, it is geometrically evident that if W be a perfect square $(\lambda x + \mu y + \nu z)^2$, this condition can only be satisfied if $\lambda x + \mu y + \nu z$ passes through one of the foci of the involution, and hence we are led to write down the following identical equation which can easily be verified:

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\lambda, \mu, \nu)^2 = -2 \begin{vmatrix} L, & M, & N \\ u_1, & u_2, & u_3 \\ v_1, & v_2, & v_3 \end{vmatrix},$$

where in u_1, &c., we are to write for x, y, z, $\mu N - \nu M$, $\nu L - \lambda N$, $\lambda M - \mu L$; that is to say, in the case we are at present considering, the determinant is

$$\begin{vmatrix} L, & M, & N, \\ \mu N - \nu M, & \nu L - \lambda N, & \lambda M - \mu L, \\ aL' + hM' + gN', & hL' + bM' + fN', & gL' + fM' + cN' \end{vmatrix}$$

where we have written L', &c., for $\mu N - \nu M$, &c. This determinant may be otherwise written

$$\begin{vmatrix} & & L, & M, & N \\ & & L', & M', & N' \\ \lambda, & L, & a, & h, & g \\ \mu, & M, & h, & b, & f \\ \nu, & N, & g, & f, & c \end{vmatrix}.$$

But in the particular case where $\lambda = \delta L = aL + hM + gN$, &c., this determinant may be reduced by subtracting the last three columns multiplied respectively by L, M, N from the first; then observing that $LL' + MM' + NN' = 0$, we see that, as we undertook to show, the determinant is divisible by $L^2 + M^2 + N^2$, the quotient being

$$\begin{vmatrix} & L' & M' & N' \\ L, & a, & h, & g \\ M, & h, & b, & f \\ N, & g, & f, & c \end{vmatrix}.$$

483. The quotient is obtained in a different and more convenient form by the following process given by Cayley. The following identities may be verified, $\mathfrak{A}$, &c., a, &c., having the meanings already explained:

$$\mathfrak{A} = \mathrm{a}\,(L^2 + M^2 + N^2) + 2L\,(N\delta M - M\delta N),$$
$$\mathfrak{B} = \mathrm{b}\,(L^2 + M^2 + N^2) + 2M\,(L\delta N - N\delta L),$$
$$\mathfrak{C} = \mathrm{c}\,(L^2 + M^2 + N^2) + 2N\,(M\delta L - L\delta M),$$
$$\mathfrak{F} = \mathrm{f}\,(L^2 + M^2 + N^2) + M\,(M\delta L - L\delta M) + N\,(L\delta N - N\delta L),$$
$$\mathfrak{G} = \mathrm{g}\,(L^2 + M^2 + N^2) + N\,(N\delta M - M\delta N) + L\,(M\delta L - L\delta M),$$
$$\mathfrak{H} = \mathrm{h}\,(L^2 + M^2 + N^2) + L\,(L\delta N - N\delta L) + M\,(N\delta M - M\delta N).$$

Hence we have

$$(\mathfrak{A}\delta L + \mathfrak{H}\delta M + \mathfrak{G}\delta N) = (\mathrm{a}\delta L + \mathrm{h}\delta M + \mathrm{g}\delta N)\,(L^2 + M^2 + N^2)$$
$$+ (L\delta L + M\delta M + N\delta N)\,(N\delta M - M\delta N),$$

with corresponding values for

$$\mathfrak{H}\delta L + \mathfrak{B}\delta M + \mathfrak{F}\delta N, \quad \mathfrak{G}\delta L + \mathfrak{F}\delta M + \mathfrak{C}\delta N,$$

and hence immediately

$$(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H})(\delta L, \delta M, \delta N)^2$$
$$= (L^2 + M^2 + N^2)\,(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(\delta L, \delta M, \delta N)^2.$$

Hence the equation, Art. 481, omitting the factor $L^2 + M^2 + N^2$, becomes

$$\mathfrak{A}\delta a + \mathfrak{B}\delta b + \mathfrak{C}\delta c + 2\mathfrak{F}\delta f + 2\mathfrak{G}\delta g + 2\mathfrak{H}\delta h$$
$$= 2\,(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})(\delta L, \delta M, \delta N)^2.$$

484. There is still another form in which the result may be expressed. Writing, as usual, in the theory of conics $bc - f^2 = A$, &c., the determinant at which we arrived at the end of Art. 482 is, when expanded,

$$-\{ALL'+BMM'+CNN'+F(MN'+M'N)$$
$$+G(NL'+N'L)+H(LM'+L'M)\}.$$

Now, from last article

$$2LL'=\mathfrak{A}-(L^2+M^2+N^2)\,\mathrm{a},\ \&c.,$$
$$MN'+M'N=\mathfrak{F}-(L^2+M^2+N^2)\,\mathrm{f},\ \&c.,$$

and remembering that $A\mathrm{a}+$ &c. $=0$, the expanded determinant last written is seen to be

$$\mathfrak{A}A+\mathfrak{B}B+\mathfrak{C}C+2\mathfrak{F}F+2\mathfrak{G}G+2\mathfrak{H}H,$$

and thus eventually the differential equation is given in the form

$$\mathfrak{A}\delta a+\mathfrak{B}\delta b+\mathfrak{C}\delta c+2\mathfrak{F}\delta f+2\mathfrak{G}\delta g+2\mathfrak{H}\delta h$$
$$=2\{\mathfrak{A}A+\mathfrak{B}B+\mathfrak{C}C+2\mathfrak{F}F+2\mathfrak{G}G+2\mathfrak{H}H\}.$$

485. As a particular case of this equation of Cayley's may be deduced that which Bouquet had given (*Liouville*, XI. p. 446) for the special case where the equation of the system of surfaces is $r=X+Y+Z$, where X, Y, Z are each functions of x, y, z respectively only. In this case then we have

$$L=X',\ M=Y',\ N=Z',\ a=X'',\ b=Y'',\ c=Z'',\ f=g=h=0;$$
$$A=Y''Z'',\ B=Z''X'',\ C=X''Y'',\ F=G=H=0;$$
$$\mathfrak{A}=(Y''-Z'')\,X'Y'Z',\ \mathfrak{B}=(Z''-X'')\,X'Y'Z',$$
$$\mathfrak{C}=(X''-Y'')\,X'Y'Z';$$
$$\delta a=X'X''',\ \delta b=Y'Y''',\ \delta c=Z'Z''',$$

and the differential equation being divisible by $X'Y'Z'$ is reduced to

$$X'X'''(Y''-Z'')+Y'Y'''(Z''-X'')+Z'Z'''(X''-Y'')$$
$$+2(Y''-Z'')(Z''-X'')(X''-Y'')=0.$$

486. Even when the equation of condition is satisfied by an assumed equation it does not seem easy to determine the two conjugate systems. Thus Bouquet observed that the condition just found is satisfied when the given system is of the form $x^l y^m z^n=r$, but he gave no clue to the discovery of the conjugate systems. This lacuna was completely supplied by Serret, who has shown much ingenuity and analytical power in deducing the equations of the conjugate systems, when

the equation of condition is satisfied.* The actual results are, however, of a rather complicated character. We must content ourselves with referring the reader to his memoir, only mentioning the two simplest cases obtained by him, and which there is no difficulty in verifying *à posteriori*. He has shown that the three equations,

$$\frac{yz}{x} = r,$$
$$\sqrt{(x^2+y^2)} + \sqrt{(x^2+z^2)} = p,$$
$$\sqrt{(x^2+y^2)} - \sqrt{(x^2+z^2)} = q,$$

represent a triple system of mutually orthogonal surfaces. The surfaces (r) are hyperbolic paraboloids. The system (p) is composed of the closed portions, and the system (q) of the infinite sheets, of the surfaces of the fourth order,

$$(z^2 - y^2)^2 - 2p^2 (z^2 + y^2 + 2x^2) + p^4 = 0.$$

Serret has observed that it follows at once from what has been stated above, that in a hyperbolic paraboloid, of which the principal parabolas are equal, the sum or difference of the distances of every point of the same line of curvature from two fixed generatrices is constant.

He finds also (in a somewhat less simple form) the following equations for another system of orthogonal surfaces,

$$p = xyz,$$
$$q = (x^2 + \omega y^2 + \omega^2 z^2)^{\frac{3}{2}} + (x^2 + \omega^2 y^2 + \omega z^2)^{\frac{3}{2}},$$
$$r = (x^2 + \omega y^2 + \omega^2 z^2)^{\frac{3}{2}} - (x^2 + \omega^2 y^2 + \omega z^2)^{\frac{3}{2}},$$

where ω is a cube root of unity.

[In continuation of the work of Serret, Darboux shows that the two families associated with the family $x^l y^m z^n = p$ are obtained by eliminating λ between the equation

$$\left(\lambda + \frac{x^2}{l}\right)^l \left(\lambda + \frac{y^2}{m}\right)^m \left(\lambda + \frac{z^2}{n}\right)^n = \text{constant}$$

and its derivative with regard to λ (Darboux, *Systèmes Orthogonaux*, Livre I., chap. vi.).]

An interesting system of orthogonal surfaces, and very

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* [The subject is dealt with by Darboux (see below), and also by Forsyth, *Differential Geometry*, p. 451.]

analogous to the system of confocal quadric surfaces, is given by Darboux in his Memoir referred to (Art. 478, note), namely, the system of cyclides

$$(x^2+y^2+z^2)^2+\frac{4d^2+a\lambda}{a+\lambda}x^2+\frac{4d^2+b\lambda}{b+\lambda}y^2+\frac{4d^2+c\lambda}{c+\lambda}z^2+d^2=0,$$

where a, b, c, d are given constants, and in place of λ we are to write successively the three parameters p, q, r. The formulæ for x, y, z in terms of p, q, r, are

$$(a^2-4d^2)\,x^2=\frac{M\,(a+p)\,(a+q)\,(a+r)}{(a-b)\,(a-c)},$$

$$(b^2-4d^2)\,y^2=\frac{M\,(b+p)\,(b+q)\,(b+r)}{(b-c)\,(b-a)},$$

$$(c^2-4d^2)\,z^2=\frac{M\,(c+p)\,(c+q)\,(c+r)}{(c-a)\,(c-b)},$$

where, writing for shortness,

$$m=\frac{(2d+p)\,(2d+q)\,(2d+r)}{4d\,(2d-a)\,(2d-b)\,(2d-c)},\quad n=\frac{(2d-p)\,(2d-q)\,(2d-r)}{4d\,(2d+a)\,(2d+b)\,(2d+c)},$$

we put $$M=\frac{4d^2}{\{\sqrt{(4dm)}\pm\sqrt{4dn}\}^2}.$$

If $d=\infty$, the system of surfaces is

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}+\tfrac{1}{4}=0,$$

which is in effect the system of confocal quadrics: a slight change of notation would make the constant term become -1.

[A very general system of surfaces discussed by Darboux (*Systèmes Orthogonaux*) consists of the three families of surfaces which are the envelopes, for all values of l, m, n, t, of the surfaces represented by the equations

$$p=\left(\lambda+\frac{x^2}{l}\right)^l\left(\lambda+\frac{y^2}{m}\right)^m\left(\lambda+\frac{z^2}{n}\right)^n\lambda^t$$

Through any point three surfaces of the family may be drawn, corresponding to the three values of λ determined by the equation $\dfrac{dp}{d\lambda}=0$, namely,

$$\frac{l}{\lambda+\frac{x^2}{l}}+\frac{m}{\lambda+\frac{y^2}{m}}+\frac{n}{\lambda+\frac{z^2}{n}}+\frac{t}{\lambda}=0.$$

If λ_1, λ_2 are two roots of this equation the condition of orthogonality is

$$\frac{x^2}{\left(\lambda_1+\frac{x^2}{l}\right)\left(\lambda_2+\frac{x^2}{l}\right)}+\frac{y^2}{\left(\lambda_1+\frac{y^2}{m}\right)\left(\lambda_2+\frac{y^2}{m}\right)}+\frac{z^2}{\left(\lambda_1+\frac{z^2}{n}\right)\left(\lambda_2+\frac{z^2}{n}\right)}=0$$

which is identically satisfied, as may be verified by eliminating t between the two equations which express that λ_1 and λ_2 are roots of the cubic.]

W. Roberts, expressing in elliptic coordinates the condition that two surfaces should cut orthogonally, has sought for systems orthogonal to $L+M+N=r$, where L, M, N are functions of the three elliptic coordinates respectively. He has thus added some systems of orthogonal surfaces to those previously known.* Of these perhaps the most interesting, geometrically, is that whose equation in elliptic coordinates is $\mu\nu=a\lambda$, and for it he has given the following construction :—Let a fixed point in the line of one of the axes of a system of confocal ellipsoids be made the vertex of a series of cones circumscribed to them. The locus of the curves of contact will be a determinate surface, and if we suppose the vertex of the cones to move along the axis, we obtain a family of surfaces involving a parameter. Two other systems are obtained by taking points situated on the other axes as vertices of circumscribing cones. The surfaces belonging to these three systems will intersect, two by two, at right angles.

It may be readily shown that the lines of curvature of the above-mentioned surfaces (which are of the third order) are circles,† whose planes are perpendicular to the principal planes of the ellipsoids. Let A, B be two fixed points, taken respectively upon two of the axes of the confocal system. To these points two surfaces intersecting at right angles will correspond, and the curve of their intersection will be the locus of points M on the confocal ellipsoids, the tangent planes at which pass through the line AB. Let P be the point where

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* *Comptes rendus*, 53, 1861, and *Journal für Math.*, 62, 1863.

† Thus they are special forms of Dupin cyclides (Art. 457*d*).

the normal to one of the ellipsoids at M meets the principal plane containing the line AB, and because P is the pole of AB in reference to the focal conic in this plane, P is a given point. Hence the locus of M, or a line of curvature, is a circle in a plane perpendicular to the principal plane containing AB.

[The following are simple examples of triply orthogonal systems.

Ex. 1. A family of parallel surfaces forms one of a triply orthogonal system, the other two families consisting of the developables generated by normals along a line of curvature.

Ex. 2. By inversion of the preceding system we see that every surface belongs to a triply orthogonal system, two of the families consisting of surfaces whose lines of curvature of one system are circles passing through any chosen point.

In the case of a Dupin cyclide (Art. 457d) the parallel surfaces are Dupin cyclides and the developables of Ex. 1 become right cones having their vertices on the focal conics, and by inverting this system we can construct any number of *triply orthogonal systems consisting of Dupin cyclides.*]

[486a. *Lamé's Curvilinear Coordinates.* If p, q, r are three independent functions of xyz, we have in general

$$x=\phi(p, q, r),\quad y=\psi(p, q, r),\quad z=\chi(p, q, r)$$

and p, q, r are said to be curvilinear coordinates of the point x, y, z. It is usual to assume with Lamé (cf. Art. 479) that the surfaces represented by $p=a$, $q=b$, $r=c$ are a triply orthogonal system. Elliptic coordinates (Arts. 409, 421a) illustrate this mode of representation.

If ds is the element of any arc in space, we easily find

$$ds^2=H^2dp^2+K^2dq^2+L^2dr^2$$

where $H^2=\left(\dfrac{dx}{dp}\right)^2+\left(\dfrac{dy}{dp}\right)^2+\left(\dfrac{dz}{dp}\right)^2=x_1^2+y_1^2+z_1^2$

with similar values for K and L, the suffixes 1, 2, 3 denoting differentiation with regard to p, q, r respectively; the cosine of the angle between two elements ds and ds' is

$$\frac{Hdpdp'+Kdqdq'+Ldrdr'}{\sqrt{(H^2dp^2+K^2dq^2+L^2dr^2)(H^2dp'^2+K^2dq'^2+L^2dr'^2)}}.$$

The direction-cosines of the tangent lines to the curve $q=\text{constant}$, $r=\text{constant}$ and the two orthogonal curves are

$$\frac{x_1}{H}, \frac{y_1}{H}, \frac{z_1}{H}; \frac{x_2}{K}, \frac{y_2}{K}, \frac{z_2}{K}; \frac{x_3}{L}, \frac{y_3}{L}, \frac{z_3}{L},$$

and thus we have six equations of the type

$$\frac{x_1^2}{H^2}+\frac{x_2^2}{K^2}+\frac{x_3^2}{L^2}=1$$

$$\frac{x_1y_1}{H^2}+\frac{x_2y_2}{K^2}+\frac{x_3y_3}{L^2}=0.$$

If we use the conditions of orthogonality $x_1x_2+y_1y_2+z_1z_2=0$ with two similar equations, it is easy to prove

$$\begin{aligned} x_1x_{11}+y_1y_{11}+z_1z_{11}&=HH_1 \\ x_2x_{11}+y_2y_{11}+z_2z_{11}&=-HH_2 \\ x_3x_{11}+y_3y_{11}+z_3z_{11}&=-HH_3. \end{aligned}$$

Multiplying in turn by $\frac{x_1}{H^2}, \frac{x_2}{K^2}, \frac{x_3}{L^2}$ and adding we find

$$x_{11}=x_1\frac{H_1}{H}-x_2\frac{HH_2}{K^2}-x_3\frac{HH_3}{L^2} \quad . \quad . \quad (1)$$

The same equation is satisfied by the differentials of y and z, and we obtain altogether nine equations by interchanging the numerals 1, 2, 3 and the corresponding magnitudes H, K, L.

Differentiating with regard to x, y, z the three expressions of the type $x_2x_3+y_2y_3+z_2z_3$, and equating the results to zero we find

$$x_1x_{23}+y_1y_{23}+z_1z_{23}=0.$$

Also

$$\begin{aligned} x_2x_{23}+y_2y_{23}+z_2z_{23}&=KK_3 \\ x_3x_{23}+y_3y_{23}+z_3z_{23}&=LL_2 \end{aligned}$$

and as before we find three equations of the form

$$x_{23}=x_2\frac{K_3}{K}+x_3\frac{L_2}{L} \quad . \quad . \quad . \quad . \quad (2)$$

which are satisfied by x, y, and z, with six other equations obtained by cyclical interchange, which become, when p; q, or r are constants, the differential equations of the lines of curvature on each surface (cf. Art. 479).

If we now substitute for the second differentials their

values given by (1) and (2) in terms of the first differentials, in the identities

$$\frac{dx_{11}}{dq}=\frac{dx_{12}}{dp}, \quad \frac{dy_{11}}{dq}=\frac{dy_{12}}{dp}, \quad \frac{dz_{11}}{dq}=\frac{dz_{12}}{dp}$$

we reach three equations

$$Rx_2+P'x_3=0, \; Ry_2+P'y_3=0, \; Rz_2+P'z_3=0$$

where P, Q, R, P', Q', R' contain only H, K, L and their differentials with regard to p, q, and r. We thus arrive at Lamé's equations of which the first three are represented by

$$P\equiv \frac{d}{dq}\left(\frac{1}{K}\frac{dL}{dq}\right)+\frac{d}{dr}\left(\frac{1}{L}\frac{dK}{dr}\right)+\frac{1}{H^2}\frac{dK}{dp}\frac{dL}{dp}=0 \quad . \quad . \; (3)$$

and the second three by

$$P'\equiv \frac{d^2H}{dqdr}-\frac{1}{K}\frac{dK}{dr}\frac{dH}{dq}-\frac{1}{L}\frac{dL}{dq}\frac{dH}{dr}=0 \quad . \quad . \; (4)$$

These six equations must be satisfied by H, K, L in order that ds^2 may be expressible in the form given, and it can be shown further* that functions H, K, L satisfying the equations determine a triply orthogonal system except as to its position and orientation in space.†

It is worthy of remark that the last written equation of Lamé may be used to deduce one form of the differential equation (Arts. 480 *sqq.*) which expresses the condition that the surfaces represented by p = constant may belong to a triply orthogonal system. The equation (3) may be written

$$\frac{1}{K^2}\frac{d}{dq}\left(\frac{1}{L^2}\frac{dH}{dr}\right)+\frac{1}{L^2}\frac{d}{dr}\left(\frac{1}{K^2}\frac{dH}{dq}\right)=0.$$

For convenience let us represent H, regarded as a function of x, y, z, by T. It is easy to prove the relation

$$H\equiv T=\left\{\left(\frac{dp}{dx}\right)^2+\left(\frac{dp}{dy}\right)^2+\left(\frac{dp}{dz}\right)^2\right\}^{-\frac{1}{2}}.$$

Lamé's equation is then equivalent to

$$(\delta_2\delta_3+\delta_3\delta_2)\,T=0$$

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* See Forsyth, *Differential Geometry*, Art. 251.

† For applications of Lamé's equation to determine particular triply orthogonal systems, see Darboux, *Systèmes Orthogonaux*, Bk. II, which deals with curvilinear coordinates.

where $$\delta_1 = \frac{1}{H^2}\frac{d}{dp} = p_1\frac{d}{dx} + p_2\frac{d}{dy} + p_3\frac{d}{dz}$$

the suffixes of p, q, r, now representing differentiation with regard to x, y, z. It may be shown without difficulty that not only T but also p, $x^2+y^2+z^2$, px, py, and pz, satisfy the equation

$$(\delta_2\delta_3 + \delta_3\delta_2)\ \phi = 0$$

which may be written, after using the conditions of orthogonality, in the form

$$\left(q_1r_1,\ q_2r_2,\ q_3r_3,\ q_2r_3+q_3r_2,\ q_3r_1+q_1r_3,\ q_1r_2+q_2r_1\right)\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)^2 \phi = 0.$$

We have thus six linear equations in the coefficients of this operator, and by eliminating them, after a slight linear transformation we obtain Cayley's equation

$$\begin{vmatrix} T_{11} & T_{22} & T_{33} & T_{23} & T_{31} & T_{12} \\ p_{11} & p_{22} & p_{33} & p_{23} & p_{31} & p_{12} \\ 2p_1 & 0 & 0 & 0 & p_3 & p_2 \\ 0 & 2p_2 & 0 & p_3 & 0 & p_1 \\ 0 & 0 & 2p_3 & p_2 & p_1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Darboux uses this equation freely, and observes that it is satisfied by any function p satisfying the equation

$$T = (x^2+y^2+z^2)\ \lambda + \alpha x + \beta y + \gamma z + \delta$$

where λ, α, β, γ, δ are arbitrary functions of p.*]

[486*b*. There is an interesting theorem on triply orthogonal surfaces, due to Darboux, who describes it as the "réciproque" of Dupin's. The two theorems, Darboux shows, may be collectively expressed as follows: *If all the surfaces of one singly infinite family of surfaces be cut orthogonally by all those of a second, the necessary and sufficient condition that the two families should belong to a triply orthogonal system is that the curve of intersection of any surface of the one family*

* For further deductions see Darboux, *Systèmes Orthogonaux.*

*with any surface of the other should always be a line of curvature on either.**

Darboux' contribution consists in showing that these conditions are *sufficient*. This amounts to proving that if l, m, n be the direction-cosines of the common tangent line at x, y, z to any two surfaces, one from each of the two families, then, from the fact that the surfaces are always orthogonal and intersect in a line of curvature, it will follow that the equation

$$ldx + mdy + ndz = 0$$

satisfies the condition of integrability † and thus represents a singly infinite family of surfaces $W =$ constant, which cut the other surfaces orthogonally. Instead of proving this theorem separately we shall indicate a method of showing that Dupin's, Darboux', and Joachimsthal's theorems ‡ are special cases of much wider relations connecting the torsions of systems of curves, each system (S) being defined by the property that the direction-cosines (l, m, n) of the principal normal at any point (P, x, y, z) in space, are given functions of x, y, z. It is clear that the system of geodesics on a singly infinite family of surfaces, $U =$ constant, is a special case of those considered, and arises when the equation

$$ldx + mdy + ndz = 0$$

is integrable in the form $U =$ constant, that is when

$$I \equiv l\left(\frac{dn}{dy} - \frac{dm}{dz}\right) + m\left(\frac{dl}{dz} - \frac{dn}{dx}\right) + n\left(\frac{dm}{dx} - \frac{dl}{dy}\right) = 0.$$

In general if PN be the principal normal (l, m, n) belonging to the system S, it may be proved by the Frenet-Serret formulæ (Art. 368a), that one and only one curve of S can be drawn through P in any chosen direction perpendicular to PN, and if $\frac{1}{\tau}$ and $\frac{1}{\tau'}$ are the torsions of two mutually orthogonal S-curves through P, then

* See Darboux, *Leçons sur les Systèmes Orthogonaux* (Paris, 1910).

† Forsyth's *Differential Equations*, Art. 152.

‡ The second theorem proved in Art. 304, p. 310.

$$\frac{1}{\tau}+\frac{1}{\tau'}=I \quad . \quad . \quad . \quad . \quad (1)$$

Now suppose we have two complexes * of this kind, S_1 and S_2, defined by the direction-cosines l_1, m_1, n_1 and l_2, m_2, n_2. Let C_1 and C_2 be the curves, one from each complex, having a common tangent line at P, whose direction-cosines are therefore $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$. Let $\frac{1}{\tau_{12}}$ be the torsion of C_1 and $\frac{1}{\tau_{21}}$ that of C_2; let θ_{12} represent the angle between the principal normals l_1, m_1, n_1 and l_2, m_2, n_2; and let s_{12} represent the arc of the curve through P whose tangent line is the common tangent line just mentioned. Then by using the Frenet-Serret formulæ we can prove

$$\frac{1}{\tau_{12}}-\frac{1}{\tau_{21}}=\pm\frac{d\theta_{12}}{ds_{12}} \quad . \quad . \quad . \quad (2)$$

Dupin's theorem is a special case of the relations (1) and (2) when applied to three systems S_1, S_2, S_3. For suppose the condition of integration is satisfied for each of these families, namely, $I_1=0$, $I_2=0$, $I_3=0$ where the suffixes denote the values of I for each system; and suppose further that $\theta_{23}=\theta_{31}=\theta_{12}=\frac{\pi}{2}$. Thus the three complexes represent the systems of geodesics on a triply orthogonal system of surfaces. Then the relation (2) gives the three equations

$$\frac{1}{\tau_{23}}-\frac{1}{\tau_{32}}=0, \quad \frac{1}{\tau_{31}}-\frac{1}{\tau_{13}}=0, \quad \frac{1}{\tau_{12}}-\frac{1}{\tau_{21}}=0,$$

and the relation (1) gives the three equations

$$\frac{1}{\tau_{12}}+\frac{1}{\tau_{13}}=0, \quad \frac{1}{\tau_{23}}+\frac{1}{\tau_{21}}=0, \quad \frac{1}{\tau_{31}}+\frac{1}{\tau_{32}}=0.$$

Hence these six torsions vanish and therefore the corresponding curves, which are clearly lines of intersection of

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* In general when only one curve of a system passes through each point in space the curves are said to form a *congruence* (extending the sense of the word used in Art. 453). In like manner the curves of one of our systems S may be said to form a *complex*, since all those passing through a given point lie on a surface associated with the point.

surfaces of different families, are lines of curvature (Art. 396*b*, Cor. 1).

Darboux' theorem also follows very simply from the two fundamental relations. For if S_1, S_2 consist of geodesics on two mutually orthogonal families of surfaces, we have $I_1=0$, $I_2=0$ and θ_{12} is a right angle everywhere ; and if S_3 represents the complex of curves whose principal normals are the tangent lines to the mutual intersections of the surfaces of these families (one from each family), then θ_{23} and θ_{31} are right angles. Hence the first five of the preceding six equations hold good and the sixth is replaced by

$$\frac{1}{\tau_{31}}+\frac{1}{\tau_{32}}=I_3.$$

The theorem now amounts to proving, what is algebraically obvious, that if $\frac{1}{\tau_{12}}=0$, i.e. if the mutual intersections of curves of the two families of surfaces are lines of curvature on the first, then we must also have

$$I_3=0,$$

and therefore S_3 is a complex of geodesics on a singly infinite family of surfaces which are clearly orthogonal to the surfaces of the other two families.] *

[486*c*. The curvature and torsion of the curves at a point P of a complex S may be investigated in the same manner as in the case of geodesics. The Frenet-Serret formulæ give

$$\frac{1}{\rho}=-\left(\alpha\frac{dl}{ds}+\beta\frac{dm}{ds}+\gamma\frac{dn}{ds}\right);\ \frac{1}{\tau}=-\begin{vmatrix}\alpha & \beta & \gamma\\ l & m & n\\ \frac{dl}{ds} & \frac{dm}{ds} & \frac{dn}{ds}\end{vmatrix}.$$

Since l, m, n are functions of x, y, z we have

$$\frac{dl}{ds}=\alpha\frac{dl}{dx}+\beta.\frac{dl}{dy}+\gamma\frac{dl}{dz};\ \text{etc.}$$

* For detailed proofs of the above relations and for other connected theorems see R. A. P. Rogers, *Some Differential Properties of the Orthogonal Trajectories of a Congruence of Curves, with an application to Curl and Divergence of Vectors*, Proceedings of the Royal Irish Acedemy, 29, A, 6 (1912).

If the point P be taken for origin and the axis of z for the normal l, m, n, we have, neglecting higher powers of x, y, z, and remembering that $ldl+mdm+ndn=0$,

$$l=a_1x+a_2y+a_3z, \quad m=b_1x+b_2y+b_3z, \quad n=1,$$

and by choosing suitable axes of x and y we can make $a_2+b_1=0$, and therefore $I=\frac{dm}{dx}-\frac{dl}{dy}=2b_1=-2a_2$. If we now put $\alpha=\cos\theta$, $\beta=\sin\theta$, $\gamma=0$, the expressions for ρ and τ become

$$\frac{1}{\rho}=\frac{\cos^2\theta}{\rho_1}+\frac{\sin^2\theta}{\rho_2}, \quad \frac{1}{\tau}=\frac{1}{2}I+\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\sin\theta\cos\theta \qquad (1)$$

ρ_1 and ρ_2 are the two extreme radii of curvature and their directions are at right angles. When these directions always coincide with those of zero torsion (which are not otherwise at right angles) we have $I=0$, and the curves are geodesics on a one-parameter system of surfaces.*

The curves whose tangent lines are the directions of extreme curvature form two "congruences" mutually orthogonal. From the equations (1) combined with the relations given in Art. 486a we can now deduce without difficulty the following generalisation of the theorems of Dupin and Darboux:—*If* S_1, S_2, S_3 *are three complexes of curves whose normals at each point are mutually orthogonal, the necessary and sufficient condition that the direction of a curve common to* S_1 *and* S_2 *should be a direction of extreme curvature on* S_1 *is* $I_2=I_3$. We have thus another proof of the theorems mentioned which assert that when $I_1=0$ and $I_2=0$ we must have $I_3=0$.]

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* It may be mentioned that when the orthogonal trajectories of the curves of the complex S form a *rectilinear* congruence (the condition for which is given in Ex. 2, Art. 455p), the value of $\frac{1}{\tau}$ for any direction is the parameter of distribution of the corresponding ruled surface and the second of the equations (1), which may evidently be reduced to the form $\frac{1}{\tau}=\frac{\cos^2\phi}{\tau_1}+\frac{\sin^2\phi}{\tau_2}$ is equivalent to the relation (Art. 462a) $\pi=\pi_1\cos^2\phi+\pi_2\sin^2\phi$.

[486*d*. *Normal Curve Congruences.* A family of curves is described as a congruence * when one passes through each point of space; l, m, n being functions of x, y, z the congruence is defined by

$$\frac{dx}{l}=\frac{dy}{m}=\frac{dz}{n}$$

or by two equations of the form

$$f_1(x, y, z, a, b)=0, \quad f_2(x, y, z, a, b)=0$$

where a and b are constant for each curve.

When the condition

$$l\left(\frac{dn}{dy}-\frac{dm}{dz}\right)+m\left(\frac{dl}{dz}-\frac{dn}{dx}\right)+n\left(\frac{dm}{dx}-\frac{dl}{dy}\right)=0$$

is always satisfied the curves are orthogonal to a singly infinite system of surfaces, and the congruence is said to be *normal.*

Another mode of representation is by means of three parameters p, q, r. The coordinates of any point in space satisfy

$$x=\phi(p, q, r),\ y=\psi(p, q, r),\ z=\chi(p, q, r) \quad . \quad (1)$$

p and q being constant the point describes a curve, and the entire assemblage of such curves for different constant values of p and q forms a congruence. We may evidently regard p, q as the coordinates of a point on the director surface, or surface of reference (cf. Art. 455*p*). The direction-cosines of any curve are clearly proportional to $\frac{dx}{dr}, \frac{dy}{dr}, \frac{dz}{dr}$, and thus the curves of the congruence are orthogonal trajectories of the curves satisfying

$$\frac{dx}{dr}dx+\frac{dy}{dr}dy+\frac{dz}{dr}dz=0.$$

Replacing dx by $\frac{dx}{dp}dp+\frac{dx}{dq}dq+\frac{dx}{dr}dr$, and dy and dz by the corresponding values, the preceding equation takes the form

$$Pdp+Qdq+Rdr=0 \quad . \quad . \quad . \quad (2)$$

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* Darboux, *Surfaces*, vol. II. Ch. I. Lilienthal, *Über die Krummung der Curvenschaaren* (Math. Ann., 32, 1888). Ribaucour, *Mémoire sur la théorie générale des surfaces courbes* (Journal des Math., IV. 7, 1891). Eisenhart, *Congruences of Curves* (Trans. Amer. Math. Soc., 4, 1903). See also notes Art. 483*b*.

where P, Q, and R are each functions of p, q, r. We can thus express *the condition that the congruence may be the orthogonal trajectories of a singly infinite family of surfaces, namely*

$$P\left(\frac{dQ}{dr}-\frac{dR}{dq}\right)+Q\left(\frac{dR}{dp}-\frac{dP}{dr}\right)+R\left(\frac{dP}{dq}-\frac{dQ}{dp}\right)=0\,. \qquad (3)$$

We can use the condition just given to prove a beautiful theorem due to Ribaucour* relating to normal congruences consisting of plane curves. In the first place it is clear that the planes of the curves being a doubly infinite system envelope a surface. The congruence may be said to be *deformed* when its envelope is deformed without stretching and the tangent planes are carried along with it so that the curves preserve their positions relative to the deformed lines on the surface. The theorem then is that *a normal congruence of plane curves continues to be normal after deformation.*

Take the envelope of the planes as the surface of reference. Let A (x, y, z) be the point of contact of the plane of a curve and let B (ξ, η, ζ) be any point on the curve. If $AB=r$ and its direction cosines are l, m, n, we have since the curve lies in the tangent plane

$$\xi=x+\lambda x_1+\mu x_2, \quad \eta=y+\lambda y_1+\mu y_2, \quad \zeta=z+\lambda z_1+\mu z_2$$

where x, y, z are functions of p and q and λ, μ are functions of p, q, and r. E, F, G being the coefficients at A in the square of the linear element (Art. 377) of the surface of reference, and the suffixes 1, 2, 3 denote differentiation with regard to p, q, and r, we find without difficulty

$$\begin{aligned}P=\lambda_3(1+\lambda_1)\,E+(\mu_3+\lambda_1\mu_3+\lambda_3\mu_1)\,F+\mu_1\mu_3 G\\ +\tfrac{1}{2}\lambda\lambda_3\frac{dE}{dp}+\tfrac{1}{2}\lambda_3\mu\frac{dE}{dq}+\lambda\mu_3\left(\frac{dF}{dp}-\tfrac{1}{2}\frac{dE}{dq}\right)+\tfrac{1}{2}\mu\mu_3\frac{dG}{dp},\end{aligned}$$

with a similar value for Q, and

$$R=\lambda_3^2\,E+2\lambda_3\mu_3\,F+\mu_3^2\,G.$$

Now if θ, ϕ be the angles between AB and the parametric tangent lines at A we have

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* *Mém. sur la théorie générale des surfaces courbes* (Journ. de Math., IV. 7, 1891).

$$\cos\theta = \Sigma\frac{\xi - x}{r}\,\frac{x_1}{\sqrt{E}} = \frac{\lambda E + \mu F}{r\sqrt{E}}$$

$$\text{and } \cos\phi = \frac{\lambda F + \mu G}{r\sqrt{G}}.$$

In the deformation in question θ, ϕ, and r are the same for corresponding points on the surface and curve, and if (as in Art. 390) we make the parametric lines correspond E, F, G are likewise unaltered by deformation. Thus λ, μ, and therefore P, Q, R, are unaltered, and hence the condition of integrability is unaffected, which proves the theorem.

If the point B lies on the surface $\Sigma = 0$, defined by the equation $Pdp + Qdq + Rdr = 0$, the corresponding point B' on the "deformed" congruence will lie on the surface $\Sigma' = 0$, defined by the equation $Pdp + Qdq + Rdr = 0$; for we have shown that P, Q, and R are unaltered. Now Σ and Σ' are orthogonal to their respective congruences; thus *all the points on any surface orthogonal to the original congruence are transformed into points on a surface orthogonal to the new congruence.*

When the envelope is a curve the theorem becomes: *If all the planes are tangent to a curve C, the congruence remains normal when the curve is twisted without change of curvature, the planes and curves preserving their positions relative to the corresponding points of contact, tangent lines, and osculating planes of C.*]

[486*e*. *Cyclic Systems.* A congruence of circles normal to a family of surfaces is described by Ribaucour as a *cyclic system.** In general the coordinates of any point on a circle whose centre is α, β, γ and radius ρ, may be written

$$x = \alpha + \rho\,(\lambda \cos\theta + \lambda' \sin\theta)$$
$$y = \beta + \rho\,(\mu \cos\theta + \mu' \sin\theta)$$
$$z = \gamma + \rho\,(\nu \cos\theta + \nu' \sin\theta)$$

where λ, μ, ν, λ', μ', ν' are the direction-cosines of two per-

* *Sur les systèmes cycliques* (*Comptes rendus de l'Acad. des Sciences*, 76, 1873), also Bianchi, *Giorno. di Mat.*, 21, 1883 (and *op. cit.*), and Guischard, Ann. Ec. Norm., III. 14, 15, 20 (1897).

pendicular lines in the plane of the circle; θ is then the angle a radius makes with a fixed radius in the plane. If α, β, γ, ρ, λ, μ, ν, λ', μ', ν' are functions of two parameters p and q, the preceding equations will represent as just explained a *congruence of circles* in the most general form, the parameter θ now taking the place of r in the equations (1). It will be found that the equation (2) takes the form

$$Pdp + Qdq + \rho d\theta = 0 \quad . \quad . \quad . \quad (4)$$

and the condition (3) becomes

$$I \equiv L \cos\theta + M \sin\theta + N = 0 \quad . \quad . \quad (5)$$

where L, M, N are functions of p and q alone. If a value of θ satisfying this equation be substituted in (4) it becomes say $Rdp + Sdq = 0$, and if it happens that $R = S = 0$ for all values of p and q, the congruence is normal to the surface $I = 0$. For given values of p and q not more than two values of θ satisfy the equation $I = 0$, unless $L = M = N = 0$, when every value satisfies. Thus, if for all values of p and q more than two values of θ satisfy $I = 0$, this equation must be an identity. Hence we have Ribaucour's theorem: *If the circles of a congruence are normal to more than two surfaces they constitute a cyclic system, that is, they are normal to a singly infinite family of surfaces.*

This method may also be used to prove that the congruence of circles normal to a plane and to any arbitrary surface form a cyclic system, and by inversion, *the circles normal to a sphere and to any arbitrary surface form a cyclic system.*]

[486*f*. A *cyclic congruence* is defined as the rectilinear congruence consisting of the axes of the circles of a cyclic system, that is, of the lines through the centres of the circles perpendicular to their planes. These axes must form a congruence since one corresponds to each circle, and they therefore form a doubly infinite system of lines in space. Each ray of the cyclic congruence meets two consecutive rays (Art. 457), and thus through each ray we have two developables of the congruence. Now it may be shown by using the conditions deducible from (5) of the foregoing article, viz. $L = 0$, $M = 0$,

$N=0$, that the developables of the cyclic congruence correspond to the lines of curvature of any surface orthogonal to the circle of the cyclic system.* Thus to each developable (Δ) corresponds a surface (Σ) generated by the associated circles, and this surface contains lines of curvature of all the surfaces orthogonal to the circles. Hence by Darboux' theorem (Art. 486*b*) or directly, the two singly infinite families of the type Σ, form with the original orthogonal surfaces a triply orthogonal system; in other words, *a singly infinite family of surfaces whose orthogonal trajectories are circles belongs to a triply orthogonal system* (Ribaucour).

Amongst other interesting properties of cyclic systems the following may be mentioned and can be proved by the foregoing methods:—

(1) The tangents to the lines of curvature on an orthogonal surface meet the corresponding axis in its focal points.

(2) The osculating circles of the curves of intersection of two families of a triply orthogonal system belong to a cyclic system.

(3) The circles normal to a sphere and to any surface are orthogonal to a singly infinite system of surfaces belonging to a triply orthogonal system. (Cf. Art. 486*e*.)

If the surface chosen is a cyclide of Dupin all the surfaces of the system are cyclides of Dupin.

(4) If all the planes of the circles of a cyclic system pass through a fixed point, each of the circles is orthogonal to a fixed sphere having that point as centre.

(5) Any four surfaces orthogonal to the circles of a cyclic system meet the circles in four points whose anharmonic ratio is constant.

For further information with regard to triply orthogonal systems the reader should refer to Lucien Lévy, *Sur les Systèmes de Surfaces Triplement Orthogonaux* in the *Memoires Couronnés*, publ. par l'Acad. R. de Belgique, LIV. (1896). This memoir contains an historical *résumé* of the whole subject. See in particular Darboux, *Leçons sur Les Systémes Orthogonaux* (Paris, 1910), in which this distinguished mathematician deduces a number of systems from the general differential equations.]

* For detailed proof see Ribaucour (*op. cit.*) or Bianchi, *Lezioni*, vol. II.

CHAPTER XIV.

THE WAVE SURFACE, THE CENTRO-SURFACE, PARALLEL, PEDAL, AND INVERSE SURFACES.

487. Before proceeding to surfaces of the third order we think it more simple to treat of some special surfaces, the theory of which is more closely connected with that explained in preceding chapters. We begin by defining and forming the equation of Fresnel's *Wave Surface.**

[The Wave Surface has received a great deal of attention from geometers, partly owing to its optical interest, and partly to the fact that it is a special type of Kummer's quartic surface containing sixteen nodes and sixteen double tangent planes (see Ch. XVI). Among the distinguished mathematicians who have investigated its properties we find, besides Fresnel, Cauchy, Herschel, Hamilton, MacCullagh, H. Lloyd, Plücker, Lamé, Bertrand, Cayley, Brioschi, W. Roberts, Mannheim (who has made it a special study), Darboux (who with others investigates its asymptotic lines and lines of curvature; *Comptes Rendus*, 1881, 1885), and Weber, who shows how to represent it by means of elliptic functions. Wölffing (*Bibliotheca Mathematica*, III. 3, 1902) gives a *résumé* of the previous work done on the subject. Models of the surface have been constructed by Brill, and (independently) by Cotter, whose model may be seen in the museum of the Engineering School, Trinity College, Dublin. For a more detailed bibliography see *Loria, op. cit.*]

THE WAVE SURFACE.

If a perpendicular through the centre be erected to the plane of any central section of a quadric, and on it lengths be taken equal to the axes of the section, the locus of their extremities will be a surface of two sheets, which is called the Wave Surface. Its equation is at once derived from Arts. 101, 102, where the lengths of the axes of any section are expressed in terms of the angles which a perpendicular to

* See Fresnel, *Mémoires de l'Institut*, vol. VII. p. 136, published 1827.

its plane makes with the axes of the surface. The same equation then expresses the relation which the length of a radius vector to the wave surface bears to the angles which it makes with the axes. The equation of the wave surface is therefore

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

where $r^2=x^2+y^2+z^2$. Or, multiplying out,

$$(x^2+y^2+z^2)\,(a^2x^2+b^2y^2+c^2z^2)$$
$$-\{a^2\,(b^2+c^2)\,x^2+b^2\,(c^2+a^2)\,y^2+c^2\,(a^2+b^2)\,z^2\}+a^2b^2c^2=0.$$

From the first form we see that the intersection of the wave surface by a concentric sphere is a sphero-conic.

488. The section by one of the principal planes (e.g. the plane z) breaks up into a circle and ellipse

$$(x^2+y^2-c^2)\,(a^2x^2+b^2y^2-a^2b^2).$$

This is also geometrically evident, since if we consider any section of the generating quadric through the axis of z, one of the axes of that section is equal to c, while the other axis lies in the plane xy. If, then, we erect a perpendicular to the plane of section, and on it take portions equal to each of these axes, the extremities of one portion will trace out a circle whose radius is c, while the locus of the extremities of the other portion will plainly be the principal section of the generating quadric, only turned round through 90°. *In each of the principal planes the surface has four double points;* namely, the intersection of the circle and ellipse just mentioned. If x', y' be the coordinates of one of these intersections, the tangent cone (Art. 270) at this double point has for its equation

$$4\,(xx'+yy'-c^2)\,(a^2xx'+b^2yy'-a^2b^2)+z^2\,(a^2-c^2)\,(b^2-c^2)=0.$$

The generating quadric being supposed to be an ellipsoid, it is evident that in the case of the section by the plane z, the circle whose radius is c, lies altogether within the ellipse whose axes are a, b; and in the case of the section by the plane x, the circle whose radius is a, lies altogether without the ellipse whose axes are b, c. Real double points occur

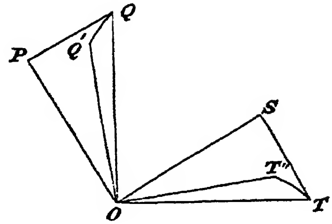
only in the section by the plane y; they are evidently the points corresponding to the circular sections of the generating ellipsoid.

The section by the plane at infinity also breaks up into factors $x^2+y^2+z^2$, $a^2x^2+b^2y^2+c^2z^2$, and may therefore also be considered as an imaginary circle and ellipse, which in like manner give rise to four imaginary double points of the surface situated at infinity. Thus *the surface has in all sixteen nodal points, only four of which are real.*

489. The wave surface is one of a class of surfaces which may be called *apsidal surfaces.* Any surface being given, if we assume any point as pole, draw any section through that pole, and on the perpendicular through the pole to the plane of section take lengths equal to the *apsidal* (that is to say, to the maximum or minimum) radii of that section; then the locus of the extremities of these perpendiculars is the apsidal surface derived from the given one. The equation of the apsidal surface may always be calculated, as in Art. 101. First form the equation of the cone whose vertex is the pole, and which passes through the intersection with the given surface of a sphere of radius r. Each edge of this cone is proved (as at Art. 102) to be an apsidal radius of the section of the surface by the tangent plane to the cone. If, then, we form the equation of the reciprocal cone, whose edges are perpendicular to the tangent planes to the first cone, we shall obtain all the points of intersection of the sphere with the apsidal surface. And by eliminating r between the equation of this latter cone and that of the sphere, we have the equation of the apsidal surface.

490. If OQ be any radius vector to the generating surface, and OP the perpendicular to the tangent plane at the point Q, then OQ will be an apsidal radius of the section passing through OQ and through OR which is supposed to be perpendicular to the plane of the paper POQ. For the tangent plane at Q passes through PQ and is perpendicular to the

plane of the paper; the tangent line to the section QOR lies in the tangent plane, and is therefore also perpendicular to the plane of the paper. Since then OQ is perpendicular to the tangent line in the section QOR, it is an apsidal radius of that section.



It follows that OT, the radius of the apsidal surface corresponding to the point Q, lies in the plane POQ, and is perpendicular and equal to OQ.

491. *The perpendicular to the tangent plane to the apsidal surface at T lies also in the plane POQ, and is perpendicular and equal to OP.**

Consider first a radius OT' of the apsidal surface, indefinitely near to OT, and lying in the plane TOR, perpendicular to the plane of the paper. Now OT' is by definition equal to an apsidal radius of the section of the original surface by a plane perpendicular to OT', and this plane must pass through OQ. Again, an apsidal radius of a section is equal to the next consecutive radius. The apsidal radius therefore of a section passing through OQ, and indefinitely near the plane QOR, will be equal to OQ. It follows, then, that $OT = OT'$, and therefore that the tangent at T to the section TOR is perpendicular to OT, and therefore perpendicular to the plane of the paper. The perpendicular to the tangent plane at T must therefore lie in the plane of the paper, but this is the first part of the theorem which was to be proved.

Secondly, consider an indefinitely near radius OT'' in the plane of the paper; this will be equal to an apsidal radius of the section ROQ', where OQ' is indefinitely near to OQ. But, as before, this apsidal radius being indefinitely near to OQ' will be equal to it, and therefore OT'' will be equal

* These theorems are due to MacCullagh, *Transactions of the Royal Irish Academy*, vol. XVI., in his collected works, p. 4, &c.

as well as perpendicular to OQ'. The angle then $T''TO$ is equal to $Q'QO$, and therefore the perpendicular OS is equal and perpendicular to OP.

It follows from the symmetry of the construction, that *if a surface A is the apsidal of B, then conversely B is the apsidal of A.*

[The preceding results may be proved analytically. Let xyz be the coordinates of an apse Q on a section of $U=0$, drawn through the origin, and let ξ, η, ζ be the coordinates of the corresponding point T on the apsidal surface. Using the condition that OQ is a limiting radius in the section it will be found that, as x, y, z satisfy the equation

$$U_1dx + U_2dy + U_3dz = 0$$

ξ, η, ζ will satisfy the equation

$$\begin{vmatrix} x & y & z \\ U_1 & U_2 & U_3 \end{vmatrix} \begin{vmatrix} d\xi & d\eta & d\zeta \\ U_1 & U_2 & U_3 \end{vmatrix} = 0$$

and the direction-cosines of the normal at T to the apsidal surface are therefore proportional to the coefficients of $d\xi$, $d\eta$, $d\zeta$ in this equation.]

492. *The polar reciprocal of an apsidal surface, with respect to the origin O, is the same as the apsidal of the reciprocal, with respect to O, of the given surface.*

For if we take on OP, OQ portions inversely proportional to them, we shall have Op, Oq, a radius vector and corresponding perpendicular on tangent plane of the reciprocal of the given surface. And if we take portions equal to these on the lines OS, OT which lie in their plane, and are respectively perpendicular to them, then, by the last article, we shall have a radius vector and corresponding perpendicular on tangent plane of the apsidal of the reciprocal. But these lengths being inversely as OS, OT are also a radius vector, and perpendicular on tangent plane of the reciprocal of the apsidal. The apsidal of the reciprocal is therefore the same as the reciprocal of the apsidal.

In particular, *the reciprocal of the wave surface generated from any ellipsoid is the wave surface generated from the reciprocal ellipsoid.*

We might have otherwise seen that the reciprocal of a wave surface is a surface also of the fourth degree, for the reciprocal of a surface of the fourth degree is in general of the thirty-sixth degree (Art. 281); but it is proved, as for plane curves, that each double point on a surface reduces the degree of its reciprocal by two; and we have proved (Art. 488) that the wave surface has sixteen double points.

To a nodal point on any surface (which is a point through which can be drawn an infinity of tangent planes, touching a cone of the second degree) answers on the reciprocal surface a tangent plane, having an infinity of points of contact, lying in a conic. From knowing then, that a wave surface has four real double points, and that the reciprocal of a wave surface is a wave surface, we infer that *the wave surface has four tangent planes which touch all along a conic.** [Further, *each of these conics is a circle.* For since the wave surface, as its equation shows, passes through the imaginary circle at infinity, every plane section passes through the I and J points in its plane, and is therefore in general a "bi-circular quartic". Hence if the plane section reduces to two coincident conics it must be a circle.]

[Ex. 1. The four real tangent planes with circular contact are represented by

$$x\sqrt{a^2-b^2} \pm z\sqrt{b^2-c^2} \pm b\sqrt{a^2-c^2} = 0.$$

Ex. 2. If the equations of the four tangent planes as just expressed are written

$$T_1 = 0,\ T_2 = 0,\ T_3 = 0,\ T_4 = 0$$

the equation of the wave surface may be written

$$T_1T_2T_3T_4 = W^2$$

where $W = a^2x^2 + b^2y^2 + c^2z^2 - a^2b^2 + b^2(x^2+y^2+z^2-a^2) + b^2(a^2-c^2)$.]

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* Sir W. R. Hamilton first showed that the wave surface has four nodes, the tangent planes at which envelope cones, and that it has four tangent planes which touch along circles. *Transactions of the Royal Irish Academy*, vol. XVI. (1837), p. 132. Dr. Lloyd experimentally verified the optical theorems thence derived, *Ibid.* p. 145. The geometrical investigations which follow are due to Professor MacCullagh, *Ibid.* p. 248. See also Plücker, "Discussion de la forme générale des ondes lumineuses," *Crelle*, t. XIX. (1839), pp. 1-44 and 91, 92.

493. We shall now prove geometrically that the four tangent planes touch along a circle. It is convenient to premise the following lemmas:

LEMMA I. "If two lines intersecting in a fixed point, and at right angles to each other, move each in a fixed plane, the plane containing the two lines envelopes a cone whose sections parallel to the fixed planes are parabolas." The plane of the paper is supposed to be parallel to one of the fixed planes and the other fixed plane is supposed to pass through the line MN. The fixed point O in which the two lines intersect is supposed to be above the paper, P being the foot of the perpendicular from it on the plane of the paper. Now let OB be one position of the line which moves in the plane OMN, then the other line OA, which is parallel to the plane of the paper being perpendicular to OB and to OP, is perpendicular to the plane OBP. But the plane OAB intersects the plane of the paper in a line BT parallel to OA, and therefore perpendicular to BP. And the envelope of BT is evidently a parabola of which P is the focus and MN the tangent at the vertex.

LEMMA II. "If a line OC be drawn perpendicular to OAB, it will generate a cone whose circular sections are parallel to the fixed planes" (Ex. 4, Art. 121). It is proved, as in Art. 125, that the locus of C is the polar reciprocal, with respect to P, of the envelope of BT. The locus is therefore a circle passing through P.

LEMMA III. "If a central radius of a quadric moves in a fixed plane, the corresponding perpendicular on a tangent plane also moves in a fixed plane." Namely, the plane perpendicular to the diameter conjugate to the first plane, to which the tangent plane must be parallel.

494. Suppose now (see figure, Art. 490) that the plane OQR (where OR is perpendicular to the plane of the paper) is a circular section of a quadric, then OT is the nodal radius of the wave surface, which remains the same while OQ moves

in the plane of the circular section; and we wish to find the cone generated by OS. But OS is perpendicular to OR which moves in the plane of the circular section and to OP which moves in a fixed plane by Lemma III, therefore OS generates a cone whose circular sections are parallel to the planes POR, QOR. Now T is a fixed point, and TS is parallel to the plane POR, therefore the locus of the point S is a circle.

The tangent cone at the node is evidently the reciprocal of the cone generated by OS, and is therefore a cone whose sections parallel to the same planes are parabolas.

Secondly, suppose the line OP to be of constant length, which will happen when the plane POR is a section perpendicular to the axis of one of the two right cylinders which circumscribe the ellipsoid, then the point S is fixed, and it is proved precisely, as in the first part of this article, that the locus of T is a circle.

495. The equations of Art. 251 give immediately another form of the equation of the wave surface. It is evident thence, that if θ, θ' be the angles which any radius vector makes with the lines to the nodes, then the lengths of the radius vector are, for one sheet,

$$\frac{1}{\rho^2} = \frac{\cos^2\frac{1}{2}(\theta - \theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta - \theta')}{a^2},$$

and for the other

$$\frac{1}{\rho'^2} = \frac{\cos^2\frac{1}{2}(\theta + \theta')}{c^2} + \frac{\sin^2\frac{1}{2}(\theta + \theta')}{a^2},$$

while $$\frac{1}{\rho^2} - \frac{1}{\rho'^2} = \left(\frac{1}{c^2} - \frac{1}{a^2}\right) \sin\theta \sin\theta'.$$

It follows hence also that *the intersections of a wave surface with a series of concentric spheres are a series of confocal sphero-conics*. For, in the preceding equations, if ρ or ρ' be constant, we have $\theta \pm \theta'$ constant.

[495*a*. The coordinates of a point on the wave surface may be expressed as *elliptic functions of two parameters* as follows:—*

* Appell and Lacour, *Fonctions Elliptiques* (1897), Art. 119.

$$\text{Let } x^2+y^2+z^2=\alpha$$
$$a^2x^2+b^2y^2+c^2z^2=\beta.$$

Then on the wave surface

$$a^2(b^2+c^2)x^2+b^2\ (c^2+a^2)y^2+c^2\ (a^2+b^2)z^2=\alpha\beta+a^2b^2c^2.$$

If we put $\quad \dfrac{\alpha-b^2}{a^2-b^2}=1-p^2, \quad \dfrac{\beta-c^2a^2}{a^2\ (b^2-c^2)}=1-q^2$

we find

$$x=bp\sqrt{1-l^2q^2}$$
$$y=a\sqrt{1-p^2}\ \sqrt{1-q^2}$$
$$z=a\sqrt{1-k^2p^2}.\, q$$
$$\text{where } k^2=\frac{a^2-b^2}{a^2-c^2} \text{ and } l^2=\frac{a^2\ (b^2-c^2)}{b^2\ (a^2-c^2)}.$$

Hence in the Jacobian notation

$$x=b\ sn\ (u,\ k)\ dn\ (v,\ l)$$
$$y=a\ cn\ (u,\ k)\ cn\ (v,\ l)$$
$$z=a\ dn\ (u,\ k)\ sn\ (v,\ l)$$

If we calculate the value of F in the linear element (Art. 377) we shall find that it vanishes, and hence the parametric curves cut at right angles. They consist of a system of sphero-quartics, $\alpha=$ constant, and of a system of quartic curves $\beta=$ constant.]

496. The equation of the wave surface has also been expressed as follows by W. Roberts in *elliptic coordinates.* The form of the equation

$$\frac{a^2x^2}{a^2-r^2}+\frac{b^2y^2}{b^2-r^2}+\frac{c^2z^2}{c^2-r^2}=0,$$

shows that the equation may be got by eliminating r^2 between the equations

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1, \text{ and } x^2+y^2+z^2=r^2.$$

Giving r^2 any series of constant values, the first equation denotes a series of confocal quadrics, the axis of z being the primary axis, and the axis of x the least; and for this system (Art. 160) $h^2=b^2-c^2$, $k^2=a^2-c^2$. Since r^2 is always less than a^2 and greater than c^2, the equation always denotes a hyper-

boloid, which will be of one or of two sheets according as r^2 is greater or less than b^2. The intersections of the hyperboloids of one sheet with corresponding spheres generate one sheet of the wave surface, and those of two sheets the other.

Now if the surface denote a hyperboloid of one sheet, and if λ, μ, ν denote the primary axes of three confocal surfaces of the system now under consideration which pass through any point, then the equation gives us $r^2 - c^2 = \mu^2$, but (Art. 161)

$$r^2 = \lambda^2 + \mu^2 + \nu^2 - h^2 - k^2,$$

whence *the equation of one sheet in elliptic coordinates is*

$$\lambda^2 + \nu^2 = c^2 + h^2 + k^2 = a^2 + b^2 - c^2.$$

In like manner *the equation of the other sheet is*

$$\lambda^2 + \mu^2 = a^2 + b^2 - c^2.$$

The general equation of the wave surface also implies $\mu^2 + \nu^2 = a^2 + b^2 - c^2$, but this denotes an imaginary locus.

Since, if λ is constant, μ is constant for one sheet and ν for the other, it follows that if through any point on the surface be drawn an ellipsoid of the same system, it will meet one sheet in a line of curvature of one system of the ellipsoid, and the other sheet in a line of curvature of the other system.

If the equations of two surfaces expressed in terms of λ, μ, ν, when differentiated give

$$Pd\lambda + Qd\mu + Rd\nu = 0, \quad P'd\lambda + Q'd\mu + R'd\nu = 0,$$

the condition that they should cut at right angles is (cf. Art. 410)

$$\frac{PP'(\lambda^2 - h^2)(\lambda^2 - k^2)}{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)} + \frac{QQ'(\mu^2 - h^2)(k^2 - \mu^2)}{(\lambda^2 - \mu^2)(\mu^2 - \nu^2)} + \frac{RR'(h^2 - \nu^2)(k^2 - \nu^2)}{(\lambda^2 - \nu^2)(\mu^2 - \nu^2)} = 0,$$

which is satisfied if $P = 0$, $Q = 0$, $R' = 0$. Hence any surface ν = constant cuts at right angles any surface whose equation is of the form $\phi(\lambda, \mu) = 0$. The hyperboloid therefore, ν = constant, cuts at right angles one sheet of the wave surface, while it meets the other in a line of curvature on the hyperboloid.

497. *The plane of any radius vector of the wave surface and the corresponding perpendicular on the tangent plane, makes equal angles with the planes through the radius vector and the nodal lines.* For the first plane is perpendicular to OR (Art. 490) which is an axis of the section QOR of the generating ellipsoid and the other two planes are perpendicular to the radii of that section whose lengths are b, the mean axis of the ellipsoid, and these two equal lines make equal angles with the axis. The planes are evidently at right angles to each other, which are drawn through any radius vector, and the perpendiculars on the tangent planes at the points where it meets the two sheets of the surface.

Reciprocating the theorem of this article, we see that the plane determined by any line through the centre and by one of the points where planes perpendicular to that line touch the surface, makes equal angles with the planes through the same line and through perpendiculars from the centre on the planes of circular contact (Art. 494).

498. If the coordinates of any point on the generating ellipsoid be $x'y'z'$, and the primary axes of confocals through that point a', a''; then the squares of the axes of the section parallel to the tangent plane are $a^2 - a'^2$, $a^2 - a''^2$, which we shall call ρ^2, ρ'^2. These, then, give the two values of the radius vector of the wave surface, whose direction-cosines are $\frac{px'}{a^2}, \frac{py'}{b^2}, \frac{pz'}{c^2}$. We shall now calculate the length and the direction-cosines of the perpendicular on the tangent plane at either of the points where this radius vector meets the surface. It was proved (Art. 491) that the required perpendicular is equal and perpendicular to the perpendicular on the tangent plane at the point where the ellipsoid is met by one of the axes of the section; and the direction-cosines of this axis are $\frac{p'x'}{a'^2}, \frac{p'y'}{b'^2}, \frac{p'z'}{c'^2}$. The coordinates of its extremity are then these several cosines multiplied by ρ, and the direction-cosines of the corresponding perpendicular of the ellipsoid are

$$P\rho\,\frac{p'x'}{a^2a'^2},\quad P\rho\,\frac{p'y'}{b^2b'^2},\quad P\rho\,\frac{p'z'}{c^2c'^2},$$

where $$\frac{1}{P^2}=\rho^2p'^2\left\{\frac{x'^2}{a^4a'^4}+\frac{y'^2}{b^4b'^4}+\frac{z'^2}{c^4c'^4}\right\}.$$

Now if the quantity within the brackets be multiplied by $(a^2-a'^2)^2$, we see at once that it will become $\frac{1}{p^2}+\frac{1}{p'^2}$. Hence

$$\frac{1}{P^2}=\frac{p^2+p'^2}{p^2\rho^2};\ \text{and}\ P^2=\frac{p^2\rho^2}{p^2+p'^2}.$$

This then gives the length of the perpendicular on the tangent plane at the point on the wave surface which we are considering. Its direction-cosines are obtained from the consideration that it is perpendicular to the two lines whose direction-cosines are respectively

$$\frac{p''x'}{a''^2},\ \frac{p''y'}{b''^2},\ \frac{p''z'}{c''^2};\qquad P\rho\,\frac{p'x'}{a^2a'^2},\ P\rho\,\frac{p'y'}{b^2b'^2},\ P\rho\,\frac{p'z'}{c^2c'^2}.$$

Forming, by Art. 15, the direction-cosines of a line perpendicular to these two, we find, after a few reductions,

$$\frac{Px'}{p\rho}\left(1-\frac{p''^2}{a''^2}\right),\quad \frac{Py'}{p\rho}\left(1-\frac{p''^2}{b''^2}\right),\quad \frac{Pz'}{p\rho}\left(1-\frac{p''^2}{c''^2}\right).$$

In fact, it is verified without difficulty, that the line whose direction-cosines have been just written is perpendicular to the two preceding.

It follows hence also, that the equation of the tangent plane at the same point is

$$xx'\left(1-\frac{p''^2}{a''^2}\right)+yy'\left(1-\frac{p''^2}{b''^2}\right)+zz'\left(1-\frac{p''^2}{c''^2}\right)=p\rho.$$

In like manner the tangent plane at the other point where the same radius vector meets the surface is

$$xx'\left(1-\frac{p'^2}{a'^2}\right)+yy'\left(1-\frac{p'^2}{b'^2}\right)+zz'\left(1-\frac{p'^2}{c'^2}\right)=p\rho'.$$

499. If θ be the angle which the perpendicular on the tangent plane makes with the radius vector, we have $P=\rho\cos\theta$; but we have, in the last article, proved $P^2=\frac{p^2\rho^2}{p^2+p'^2}$. Hence, $\cos^2\theta=\frac{p^2}{p^2+p'^2}$, $\tan^2\theta=\frac{p'^2}{p^2}$. This expression may be trans-

formed by means of the values given for p and p' (Art. 165). We have therefore

$$p^2 = \frac{a^2b^2c^2}{\rho^2\rho'^2}, \quad p'^2 = \frac{(a^2-\rho^2)(b^2-\rho^2)(c^2-\rho^2)}{\rho^2(\rho^2-\rho'^2)}.$$

Whence $$\tan^2\theta = -\frac{\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho^2}{b^2}\right)\left(1-\frac{\rho^2}{c^2}\right)}{1-\frac{\rho^2}{\rho'^2}}.$$

In this form the equation states a property of the ellipsoid, and the expression is analogous to that for the angle between the normal and central radius vector of a plane ellipse, viz.

$$\tan^2\theta = -\left(1-\frac{\rho^2}{a^2}\right)\left(1-\frac{\rho^2}{b^2}\right).$$

In the case of the wave surface it is manifest that $\tan\theta$ vanishes only when $\rho = a$, b, or c, and becomes indeterminate when $\rho = \rho' = b$.

500. The expression $\tan\theta = \frac{p'}{p}$ leads to a construction for the perpendiculars on the tangent planes at the points where a given radius vector meets the two sheets of the surface. The perpendiculars must lie in one or other of two fixed planes (Arts. 497, 498), and if a plane be drawn perpendicular to the radius vector of the wave surface at a distance p, it is evident from the expression for $\tan\theta$, that p' is the distance to the radius vector from the point where the perpendicular on the tangent plane meets this plane. Thus we have the construction, "Draw a tangent plane to the generating ellipsoid perpendicular to the given radius vector, from its point of contact let fall perpendiculars on the two planes of Art. 497, then the lines joining to the centre the feet of these perpendiculars are the perpendiculars required."

We obtain by reciprocation a similar construction, to determine the points where planes parallel to a given one touch the two sheets of the surface.

Ex. 1. To transform the equation of the surface, as at Art. 174, so as to make the radius vector to any point on the surface the axis of z, and the axes of the corresponding section of the generating ellipsoid the axes of x and y we find

$$(x^2+y^2+z^2)\{p^2z^2+(p'^2+\rho^2)x^2+(p''^2+\rho'^2)y^2+2pp'xz+2pp''yz+2p'p''xy\}$$
$$-p^2z^2(\rho^2+\rho'^2)-x^2(p^2\rho^2+p'^2\rho'^2+p''^2\rho^2+\rho^2\rho'^2)$$
$$-y^2(p^2\rho'^2+p'^2\rho'^2+p''^2\rho^2+\rho^2\rho'^2)-2pp'\rho^2xz-2pp''\rho^2yz+p^2\rho^2\rho'^2=0.$$

It is easy to see that if we make x and $y=0$ in the equation thus transformed, we get for z^2 the values ρ^2 and ρ'^2 as we ought. If we transform the equation to parallel axes through the point $z=\rho$, the linear part of the equation becomes

$$2p\rho(\rho^2-\rho'^2)(pz+p'x),$$

from which the results already obtained as to the position of the tangent plane may be independently established.

Ex. 2. To transform similarly the equation of the reciprocal of the wave surface obtained by writing $\frac{\lambda^2}{a}$ for a, &c., in the equation of the wave surface we find

$$(x^2+y^2+z^2)\{p^2\rho'^2x^2+p^2\rho^2y^2-2pp'\rho'^2xz-2pp''\rho^2yz+z^2(p'^2\rho'^2+p''^2\rho^2+\rho^2\rho'^2)\}$$
$$-\lambda^4(p^2+p''^2+\rho'^2)x^2-\lambda^4(p^2+p'^2+\rho^2)y^2-\lambda^4(p'^2+p''^2+\rho^2+\rho'^2)z^2$$
$$+2\lambda^4p'p''xy+2\lambda^4pp'xz+2\lambda^4pp''yz+\lambda^8=0.$$

We know that the surface is touched by the plane $\rho z=\lambda^2$, and if we put in this value for z, we find, as we ought, a curve having for a double point the point $y=0$, $p\rho x=p'\lambda^2$. If in the equation of the curve we make $y=0$, we get

$$\left(px-\frac{p'\lambda^2}{\rho}\right)^2\left\{\rho'^2x^2+\frac{\lambda^4}{\rho^2}(\rho'^2-\rho^2)\right\},$$

from which we learn that that chord of the outer sheet of the wave surface which joins any point on the inner sheet to the foot of the perpendicular from the centre on the tangent plane is bisected at the foot of the perpendicular. The inflexional tangents are parallel to

$$\{p'^2\rho'^2+p^2(\rho'^2-\rho^2)\}x^2-2p'p''\rho^2xy+\{p'^2\rho^2+\rho^2(\rho'^2-\rho^2)\}y^2,$$

a result of which I do not see any geometrical interpretation.

THE SURFACE OF CENTRES.

501. We have already shown (Art. 206) how to obtain the equation of the surface of centres of a quadric. We consider the problem under a somewhat more general form, as it has been discussed by Clebsch (*Crelle*, vol. LXII. p. 64), some of whose results we give, working with the canonical form; and we refer to his paper for fuller details and for his method of dealing with the general equation. By the method of Art. 227, we may consider the normal to a surface as a particular

case of the line joining the point of contact of any tangent plane to the pole of that plane with respect to a certain fixed quadric. The problem then of drawing a normal to a quadric from a given point may be generalized as follows: Let it be required to find a point $xyzw$ on a quadric $U\,(ax^2+by^2+cz^2+dw^2)$, such that the pole, with respect to another quadric V,

$$(x^2+y^2+z^2+w^2),$$

of the tangent plane to U at $xyzw$, shall lie on the line joining $xyzw$ to a given point $x'y'z'w'$. The coordinates of any point on this latter line may be written in the form

$$x'-\lambda x,\ y'-\lambda y,\ z'-\lambda z,\ w'-\lambda w,$$

and expressing that the polar plane of this point, with regard to V, shall be identical with the polar plane of $xyzw$, with respect to U, we get the equations

$$x'=(a\mu+\lambda)\,x,\ \ y'=(b\mu+\lambda)\,y,\ \ z'=(c\mu+\lambda)\,z,\ \ w'=(d\mu+\lambda)\,w.$$

And since $xyzw$ is a point on U, $\lambda:\mu$ is determined by the equation

$$\frac{ax'^2}{(a\mu+\lambda)^2}+\frac{by'^2}{(b\mu+\lambda)^2}+\frac{cz'^2}{(c\mu+\lambda)^2}+\frac{dw'^2}{(d\mu+\lambda)^2}=0.$$

When $\lambda:\mu$ is known, x, y, z, w are determined from the preceding system of equations, and since the equation in $\lambda:\mu$ is of the sixth degree, the problem admits of six solutions. If we form the discriminant, with regard to $\lambda:\mu$, of this equation, we get the locus of points $x'y'z'w'$ for which two values of $\lambda:\mu$ coincide, and rejecting a factor $x'^2y'^2z'^2w'^2$ (which indicates that two values coincide for all points on the principal planes), we shall have a surface of the twelfth degree answering to the surface of centres.

502. The problem of finding the surface of centres itself is easily made to depend on an equation of like form; for (Art. 197) the coordinates of a centre of curvature answering to any point $x'y'z'$ on an ellipsoid are

$$x=\frac{a'^2x'}{a^2},\ \ y=\frac{b'^2y'}{b^2},\ \ z=\frac{c'^2z'}{c^2}.$$

Solve for x', y', z' from these equations, and substitute in the equations satisfied by $x'y'z'$, viz.

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2}=1, \quad \frac{x'^2}{a^2a'^2}+\frac{y'^2}{b^2b'^2}+\frac{z'^2}{c^2c'^2}=0,$$

now for a'^2 write a^2-h^2, &c., and we get

$$\frac{a^2x^2}{(a^2-h^2)^2}+\frac{b^2y^2}{(b^2-h^2)^2}+\frac{c^2z^2}{(c^2-h^2)^2}=1,$$

$$\frac{a^2x^2}{(a^2-h^2)^3}+\frac{b^2y^2}{(b^2-h^2)^3}+\frac{c^2z^2}{(c^2-h^2)^3}=0.$$

These two equations represent a curve of the fourth degree, which is the locus of the centres of curvature answering to points on the intersection of the given quadric with a given confocal. The surface of centres is got by eliminating h^2 between the equations; or (since the second equation is the differential of the first with respect to h^2) by forming the discriminant of the first equation.

503. I first showed, in 1857 (*Quarterly Journal*, vol. II. p. 218), that the problem of finding the surface of centres is reducible to elimination between a cubic and a quadratic, and Clebsch has proved that the same reduction is applicable to the problem considered in its most general form. In fact, let Δ denote the discriminant of $\mu U+\lambda V$; which for the canonical form (Art. 141), is $(a\mu+\lambda)\ (b\mu+\lambda)\ (c\mu+\lambda)\ (d\mu+\lambda)$, and let Ω denote the reciprocal of $\mu U+\lambda V$, viz.

$(b\mu+\lambda)\ (c\mu+\lambda)\ (d\mu+\lambda)\ x^2+(c\mu+\lambda)\ (d\mu+\lambda)\ (a\mu+\lambda)\ y^2+$ &c.

then we have $$\frac{\Omega}{\Delta}=\frac{x^2}{a\mu+\lambda}+\frac{y^2}{b\mu+\lambda}+\text{\&c.}$$

Now, if we differentiate the right-hand side of this equation with respect to μ, we obtain the equation (Art. 501) which determines $\lambda:\mu$ which therefore may be written

$$\Omega\,\frac{d\Delta}{d\mu}=\Delta\,\frac{d\Omega}{d\mu}.$$

This last equation, which is the Jacobian of Ω and Δ, being the result of eliminating m between $\Delta+m\lambda\Omega$ and its differential,* will be verified when $\Delta+m\lambda\Omega$ has two equal factors.

* The factor λ is introduced to make $\Delta+m\lambda\Omega$ a homogeneous quartic function of $\mu:\lambda$.

Its differential again $\Omega \frac{d^2\Delta}{d\mu^2} = \Delta \frac{d^2\Omega}{d\mu^2}$ being the result of elimination of m between $\Delta + m\lambda\Omega$ and its second differential, will be verified when $\Delta + m\lambda\Omega$ has three equal factors. But both Jacobian and its differential vanish when both Δ and Ω vanish. Thus then, as was stated (Note, p. 143), the discriminant of the Jacobian of two algebraic functions Δ, Ω, contains as a factor the result of eliminating $\lambda : \mu$ between Δ and Ω; and as another factor, the condition that it shall be possible to determine m, so that $\Delta + m\lambda\Omega$ may have three equal factors. In the present case the eliminant (with respect to $\lambda : \mu$) of Δ, Ω, gives the factor $x^2y^2z^2w^2$, and it is the other condition which gives the surface answering to the surface of centres. And this condition is formed, as in Art. 206, by eliminating m between the S and T of the biquadratic $\Delta + m\lambda\Omega$.

504. The discriminant of any algebraic function

$$k\psi(\lambda) + (\lambda - a)^2\phi(\lambda),$$

must evidently be divisible by k; and if after the division we make $k = 0$, it can be proved that the remaining factor is $\psi(a)\ \phi(a)^3$ multiplied by the discriminant of $\phi(\lambda)$. Thus, then, the section of Clebsch's surface by the principal plane w is the conic $\frac{ax^2}{(a-d)^2} + \frac{by^2}{(b-d)^2} + \frac{cz^2}{(c-d)^2}$ taken three times, together with the curve of the sixth degree, which is the reduced discriminant of

$$\frac{ax^2}{(a+\lambda)^2} + \frac{by^2}{(b+\lambda)^2} + \frac{cz^2}{(c+\lambda)^2}.$$

Clebsch has remarked that this conic and curve touch each other, and the method we have adopted leads to a simple proof of this. For evidently the discriminant of

$$\frac{ax^2}{(a+\lambda)^2} + \frac{by^2}{(b+\lambda)^2} + \frac{cz^2}{(c+\lambda)^2} = 0,$$

may be regarded as the envelope of all conics which can be represented by this equation, and therefore touches every particular conic of the system in the four points where it meets the conic represented by the differential of the equation with regard to λ, viz.

$$\frac{ax^2}{(a+\lambda)^3}+\frac{by^2}{(b+\lambda)^3}+\frac{cz^2}{(c+\lambda)^3}=0.$$

The coordinates of these points are $ax^2=(a+\lambda)^3\ (b-c)$, $by^2=(b+\lambda)^3(c-a)$, $cz^2=(c+\lambda)^3(a-b)$; and the equations of the common tangents at them to the conic and its envelope are

$$x\sqrt{\left\{\frac{(b-c)a}{a+\lambda}\right\}}\pm y\sqrt{\left\{\frac{(c-a)b}{b+\lambda}\right\}}\pm z\sqrt{\left\{\frac{(a-b)c}{c+\lambda}\right\}}=0.$$

In the case under consideration $\lambda=-d$. If, then, we use the abbreviations

$$(a-b)\ (a-c)\ (a-d)=-A^2,\quad (b-a)\ (b-c)\ (b-d)=-B^2,$$
$$(c-a)\ (c-b)\ (c-d)=-C^2,\quad (d-a)\ (d-b)\ (d-c)=-D^2,$$

the equations of the common tangents to the conic and the envelope curve, are

$$\frac{xa^{\frac{1}{2}}}{A}\pm\frac{yb^{\frac{1}{2}}}{B}\pm\frac{zc^{\frac{1}{2}}}{C}=0.$$

The reasoning used in this article can evidently be applied to other similar cases. Thus, the surface parallel to a quadric (Art. 202, Ex. 2) is met by a principal plane in a curve of the eighth order and a conic, taken twice, which touches that curve in four points; and again, the four right lines (Art. 216) touch the conic in their plane.

505. Besides the cuspidal conics in the principal planes, there are other cuspidal conics on the surface, which are found by investigating the locus of points for which the equation of the sixth degree (Art. 501) has three equal roots. Differentiating that equation twice with regard to λ, we arrive at a system of equations reducible to the form

$$\frac{ax^2}{(a+\lambda)^4}+\frac{by^2}{(b+\lambda)^4}+\frac{cz^2}{(c+\lambda)^4}+\frac{dw^2}{(d+\lambda)^4}=0,$$
$$\frac{a^2x^2}{(a+\lambda)^4}+\frac{b^2y^2}{(b+\lambda)^4}+\frac{c^2z^2}{(c+\lambda)^4}+\frac{d^2w^2}{(d+\lambda)^4}=0,$$
$$\frac{a^3x^2}{(a+\lambda)^4}+\frac{b^3y^2}{(b+\lambda)^4}+\frac{c^3z^2}{(c+\lambda)^4}+\frac{d^3w^2}{(d+\lambda)^4}=0.$$

The result of eliminating λ between these three equations will be a pair of equations denoting a curve locus. Now solving these equations, we get

$$\frac{ax^2}{(a+\lambda)^4} = (b-c)(c-d)(d-b),\ \frac{by^2}{(b+\lambda)^4} = (c-a)(a-d)(c-d), \text{\&c.}$$

whence $a+\lambda$, $b+\lambda$, &c., are proportional to $a^{\frac{1}{4}}x^{\frac{1}{2}}A^{\frac{1}{2}}$, &c. Substituting from these in the equation (Art. 501)

$$\frac{ax^2}{(a+\lambda)^2} + \frac{by^2}{(b+\lambda)^2} + \frac{cz^2}{(c+\lambda)^2} + \frac{dw^2}{(d+\lambda)^2} = 0,$$

we get

$$\frac{a^{\frac{1}{2}}x}{A} \pm \frac{b^{\frac{1}{2}}y}{B} \pm \frac{c^{\frac{1}{2}}z}{C} \pm \frac{d^{\frac{1}{2}}w}{D} = 0;$$

whence we learn that the locus which we are investigating consists of curves situated in one or other of eight planes; and that these planes meet the principal planes in the common tangents to the conic and envelope curve considered in the last article.*

But if we eliminate λ between the three equations

$$a+\lambda = a^{\frac{1}{4}}x^{\frac{1}{2}}A^{\frac{1}{2}},\ b+\lambda = b^{\frac{1}{4}}y^{\frac{1}{2}}B^{\frac{1}{2}},\ c+\lambda = c^{\frac{1}{4}}z^{\frac{1}{2}}C^{\frac{1}{2}},$$

so as to form a homogeneous equation in x, y, z, we get

$$a^{\frac{1}{4}}A^{\frac{1}{2}}(b-c)\,x^{\frac{1}{2}} + b^{\frac{1}{4}}B^{\frac{1}{2}}(c-a)\,y^{\frac{1}{2}} + c^{\frac{1}{4}}C^{\frac{1}{2}}(a-b)\,z^{\frac{1}{2}} = 0,$$

which denotes a cone of the second degree touched by the planes x, y, z. Hence, the cuspidal curves in the eight planes are conics which touch the cuspidal conics in the principal planes.

506. There will be a nodal curve on the surface answering to the points for which the equation of Art. 501 has two pairs of equal roots. Now we saw (Art. 503) that the condition for a single pair of equal roots is got by eliminating m

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* The existence of these eight planes may be also inferred from the consideration that the reciprocal of the surface of centres has an equation of the form (Art. 199) $U^2 = VW$, and has therefore as double points the eight points of intersection of U, V, W. The surface of centres then has eight imaginary double tangent planes, which touch the surface in conics (see Art. 271). The origin of these planes is accounted for geometrically, as Darboux has shown, by considering the eight generators of the quadric which meet the circle at infinity (Art. 139). The normals along any of these all lie in the plane containing the generator and the tangent to the circle at infinity at the point where it meets it, and they envelope a conic in that plane. In like manner a cuspidal plane curve on the centro-surface will arise every time that a surface contains a right line which meets the circle at infinity.

between a quadratic and a cubic equation, namely, the S and T of the biquadratic $\Delta + m\lambda\Omega$. If we write these equations

$$a + bm + cm^2 = 0, \quad A + Bm + Cm^2 + Dm^3 = 0,$$

it will be found that the degrees in x, y, z, w of these coefficients are respectively 0, 2, 4; 0, 2, 4, 6; and the result of elimination is, as we know, of the twelfth degree. Now the condition that the equation of Art. 501 may have two pairs of equal roots, is simply that this cubic and quadratic may have two common values of m. Generally, if the result of eliminating an indeterminate m between two equations denotes a surface, the system of conditions that the equations shall have two common roots will represent a double curve on that surface. Thus the result of eliminating m between two quadratics $a + bm + cm^2$, $a' + b'm + c'm^2$ is

$$(ac' - ca')^2 + (ba' - ab')\ (bc' - cb') = 0.$$

But if we remember that $a\,(bc' - cb') = b\,(ac' - ca') + c\,(ba' - ab')$, this result may be written

$$a\,(ac' - ca')^2 + b\,(ac' - ca')\ (ba' - ab') + c\,(ba' - ab')^2 = 0,$$

showing that the intersection of $ac' - ca'$, $ba' - ab'$ (which must separately vanish if the equations have both roots common), is a double curve on the surface.

And to come to the case immediately under consideration, if we have to eliminate between

$$a + bm + cm^2 = 0, \quad A + Bm + Cm^2 + Dm^3 = 0,$$

we may substitute for the second equation that derived by multiplying the first by A, the second by a, and subtracting, viz.

$$(aB - bA) + (aC - cA)\ m + aDm^2 = 0,$$

and thus, as has been just shown, the result of elimination may be written $aP^2 - bPQ + cQ^2 = 0$, where

$$P = bcA - acB + a^2D, \quad Q = (ac - b^2)\ A + abB - a^2C.$$

We thus see that the curve PQ is a double curve on the surface of centres; but since P is of the sixth degree and Q of the fourth, the nodal curve PQ is of the twenty-fourth.

Further details will be found in Clebsch's paper already referred to.*

507. It is convenient to give here an investigation of some of the characteristics of the centro-surface of a surface of the m^{th} degree.† We denote by n the class of the surface, or the degree of its reciprocal, which, when the surface has no multiple points, is $m(m-1)^2$ (see Art. 281); and we denote by a the number of tangent lines to the surface which both pass through a given point and lie in a given plane, which is in the same case $m(m-1)$, Art. 282, this characteristic being the same for a surface and for its reciprocal.

Let us first examine the number of normals to a given surface (bitangents to the centro-surface, see Art. 306) which can be drawn through a given point. This is solved as the corresponding problem for plane curves. (See *Higher Plane Curves*, p. 94, and *Cambridge and Dublin Mathematical Journal*, vol. II.) Taking the point at infinity, the number of finite normals which can be drawn through it is the same as the number of tangent planes which can be drawn parallel to a given one; that is to say, is n. To this number must be added the number of normals which lie altogether at infinity. Now it is easy to see that the normal corresponding to any point of the surface at infinity lies altogether at infinity, and is the normal to the section by the plane infinity, in the extended sense of the word normal (*Higher Plane Curves*, Art. 109). The number of such normals that can be drawn through a point in the plane is $m+a$ (*Higher Plane Curves*, Art. 111), since a is the order of the reciprocal of a plane section. The total number of normals therefore that can be drawn through

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* See also a Memoir by Cayley (*Cambridge Philosophical Transactions*, vol. XII.) in which this surface is elaborately discussed. He uses the notation explained, note, Art. 409, when the equations of Art. 197 become

$$-\beta\gamma a^2x^2 = (a^2+p)^3(a^2+q),\quad -\gamma\alpha b^2y^2 = (b^2+p)^3(b^2+q),$$
$$-\alpha\beta c^2z^2 = (c^2+p)^3(c^2+q),$$

α, β, γ having the same meaning as in Art. 206.

† This investigation is derived from a communication by Darboux to the French Academy, *Comptes Rendus*, t. LXX. (1870), p. 1328.

any point is $m+n+a$; or, when the surface has no multiple points, is m^3-m^2+m.

Next let us examine the number of normals which lie in a given plane. The corresponding tangent planes evidently pass through the same point at infinity, viz. the point at infinity on a perpendicular to the given plane. And the corresponding points of contact are evidently the intersections by the given plane of the curve of contact of tangents from that point, and are therefore in number a or $m(m-1)$.

The normals to a surface constitute a congruence of lines (see Art. 453), and the two numbers just determined are the order and class of that congruence.

508. To find the locus of points on a surface, the normals at which meet a given line,

$$Ax+By+Cz+D=0,\ A'x+B'y+C'z+D'=0.$$

Substituting in these equations the values for the coordinates of a point on the normal (Art. 273), $x=x'+\theta U_1$, $y=y'+\theta U_2$, $z=z'+\theta U_3$, and eliminating the indeterminate θ, we see that the point of contact lies on the curve of intersection of the given surface with

$$(Ax+By+Cz+D)\,(A'U_1+B'U_2+C'U_3)$$
$$=(A'x+B'y+C'z+D')\,(AU_1+BU_2+CU_3),$$

a surface also of the m^{th} order, and containing the given line. The section of this curve by any plane through that line consists of the a points whose normals lie in the plane, and the m points where the line meets the surface.

509. We can hence determine the class of the centro-surface. A tangent plane to that surface contains two infinitely near normals to the given surface (Art. 306); and therefore the tangent planes to the centro-surface which pass through a given line will touch the locus determined in the last article. Now the number of planes which can be drawn to touch the curve of intersection of two surfaces of the m^{th} order, being equal to the rank of the corresponding developable, is (Arts. 325, 342) $m^2(2m-2)$; but, since in this case the line

through which the tangent planes are drawn meets the curve in m points, this number must be diminished by $2m$. The class of the centro-surface therefore is $2m(m^2 - m - 1)$.

510. Darboux* investigates as follows the degree of the centro-surface. Let μ and ν be the two numbers determined in Art. 507, viz. the order and class of the congruence formed by the normals; let M and N be the degree and class of the centro-surface.

Now take any line and consider the correspondence between two planes drawn through it such that a normal in one plane intersects a normal in the other. Drawing the first plane arbitrarily, any of the ν normals in that plane may be taken for the first normal, and at the point where it meets the arbitrary line, $\mu - 1$ other normals may be drawn; we see then that to any position of one plane correspond $\nu(\mu - 1)$ positions of the other. It follows then, from the general theory of correspondence, that there will be $2\nu(\mu - 1)$ cases of coincidence of the two planes. Now let us denote by x the number of points on the line such that the line is coplanar with two of the normals at the point; then the cases of coincidence obviously answer either to points x or to points on the centro-surface, since for each of the latter points two of the normals drawn from it coincide. We have then

$$2\nu(\mu - 1) = x + M.$$

But in like manner consider the correspondence between points on the line such that a normal from one is coplanar with a normal from the other, and we have

$$2\mu(\nu - 1) = x + N,$$

whence $$M - N = 2(\mu - \nu)$$

and putting in the values already obtained for μ, ν, N, we have

$$M = 2m(m - 1)(2m - 1).$$

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* Similar investigations were also made independently by Lothar Marcks. (See *Math. Annalen*, vol. v.) The investigation may be regarded as establishing a general relation (which seems to be due to Klein) between the order and class of an algebraic congruence, and the degree and class of its "focal surface" (see Art. 457).

511. The number thus found for the degree of the centro-surface may be verified by considering the section of that surface by the plane infinity. Consider first the section of the surface itself by the plane infinity; the corresponding normals lie at infinity, and their envelope will (*Higher Plane Curves*, Art. 112) be a curve of the degree $3a+\kappa$. And besides (as in Art. 198) the centro-surface will include the polar reciprocal of the section with regard to the circle at infinity. The degree of this will be a, and it will be counted three times. Consider now the finite points of the surface. In order that one of these should have an infinitely distant centre of curvature, two consecutive normals must be parallel, and therefore the point must be on the parabolic curve. It is easy to see that the normals along the intersection of the surface by another whose order is m', generate a surface of the degree m^2m'; therefore the normals along the parabolic curve generate a surface whose degree is $4m^2(m-2)$. But the section of this surface by the plane infinity includes the $4m(m-2)$ normals at the points where the parabolic curve itself meets the plane at infinity. The curve locus therefore at infinity answering to finite points on the parabolic curve is of the degree $4m(m-1)(m-2)$. The total degree then of the section of the centro-surface by the plane infinity, is

$$3m(m-1)+3m(m-1)+4m(m-1)(m-2),$$

or $2m(m-1)(2m-1)$ as before.

511*a*. In general 28 bitangents can be drawn to the centro-surface of a quadric from any point. In fact the reciprocals are bitangents to a plane section of the reciprocal surface which is of the fourth degree. F. Purser* has shown that these 28 lines resolve into three groups, the six normals which can be drawn from the point to the surface, the six pairs of generators of the six quadrics of the system

$$\frac{a^2x^2}{(a^2-h^2)^2}+\frac{b^2y^2}{(b^2-h^2)^2}+\frac{c^2z^2}{(c^2-h^2)^2}=1,$$

* *Quarterly Journal of Mathematics*, vol. XIII, p. 338.

which pass through the point, and the ten *synnormals* through the point. To explain what these last are; the six feet of normals from any point to a quadric may be distributed in ten ways into pairs of threes, each three determining a plane. The two planes of a pair are simply related and besides each plane touches a surface of the fourth class, or, in other words, the pole of such a plane with regard to the quadric moves on a surface of the fourth degree, to which the name *normopolar* surface has been given. The analysis which establishes this, easily shows that three intersecting normals to the quadric at points of such a plane section meet in a point which describes a definite right line when the plane section remains unaltered, which locus line corresponding to any two correlated planes satisfying the condition of the fourth order, is called a synnormal. There are therefore ten synnormals through a point.*

PARALLEL SURFACES.

512. We have discussed, Art. 202, the problem of finding the equation of a surface parallel to a quadric, and we investigate now the characteristics of the parallel to a surface of the n^{th} degree. We confine ourselves to the case when the surface has no special relation to the plane or circle at infinity. The same principles are used as in the corresponding investigation for plane curves (*Higher Plane Curves*, p. 101). The degree of the parallel is found by making k the modulus $=0$ in its equation, which will not affect the terms of highest degree in the equation. The result will represent the original surface counted twice, together

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* In 1862 Desboves published his "Théorie nouvelle des normales aux surfaces du second ordre," in which the locus line and the related surface are discussed under the names synnormal and normopolar surface. Purser independently arrived at the same results (*Quarterly Journal*, vol. VIII. p. 66) and showed the equivalence of the relation of the fourth order with the invariant relation *in plano* that three feet of normals from a point to a quadric form a triangle inscribed in one and circumscribed to another given conic; and gave a construction for any synnormal through a point.

with the developable enveloped by the tangent planes * to the surface drawn through the tangent lines of the circle at infinity, this developable answering to the tangents from the foci of a plane curve (Art. 146). Now it will be seen (Chap. XVII. *post*) that the rank of a developable enveloping a surface and a curve is $nm' + ar'$, where a, n are characteristics of the surface and m', r' of the curve. In the present case $m' = r' = 2$, and the rank of the developable is $2(n + a)$. The degree of the parallel surface is therefore $2(m + n + a)$ or $2(m^3 - m^2 + m)$; in other words, it is double the number of normals that can be drawn from a point to the surface (Art. 507).

513. If the equation of the tangent plane to a surface be $\alpha x + \beta y + \gamma z + \delta = 0$, and if the surface be given by a tangential equation between $\alpha, \beta, \gamma, \delta$, then the corresponding equation of a parallel surface is got by writing in this equation for δ, $\delta + k\rho$, where $\rho^2 = \alpha^2 + \beta^2 + \gamma^2$. This equation cleared of radicals will ordinarily be of double the degree of the primitive equation; hence the class of a parallel is in general double the class of the primitive. More generally, to a cylinder enveloping the primitive corresponds a cylinder enveloping the parallel surface, and being the parallel of the former cylinder. Hence the characteristics of the general tangent cone to the parallel are derived from those of the general tangent cone to the primitive by the rules for plane curves (*Higher Plane Curves*, Art. 117*a*). Thus then, since (Art. 279 *et seq.*) we have for the tangent cone to the primitive,

$$\mu = a = m(m-1), \qquad \nu = n = m(m-1)^2,$$
$$\kappa = 3m(m-1)(m-2), \qquad \iota = 4m(m-1)(m-2),$$

we have for the tangent cone to the parallel (*Higher Plane Curves*, l. c.)

$$\mu = 2(n + a) = 2m^2(m-1), \qquad \nu = 2n,$$
$$\kappa = 2m(m-1)(4m-5), \qquad \iota = 8m(m-1)(m-2).$$

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* It is to be noted that every parallel to any of these planes coincides with the plane itself. The paper of Mr. S. Roberts which I use in this article is in *Proceedings of the London Mathematical Society*, 1873.

Again, the reciprocal of a parallel surface is of the degree $2n$, having a cuspidal curve of the order $8m(m-1)(m-2)$, and a nodal of the order

$$m(m-1)(2m^4-6m^3+6m^2-16m+25).$$

The parallel surface will ordinarily have nodal and cuspidal curves. In fact, since the equation of the parallel surface may also be regarded as an equation determining the lengths of the normals from any point to the surface, if we form the discriminant of this with regard to k (see *Conics*, p. 337), it will include a factor which will represent a surface locus, from each point of which two distinct normals of equal length can be drawn to the surface. Such a point will be a double point on the parallel surface whose modulus is equal to this length. In like manner, each parallel surface will have a determinate number of triple points. The discriminant just mentioned will also include a factor representing the surface of centres; and plainly to those points on the primitive at which a principal radius of curvature is equal to the modulus, will correspond points on the surface of centres which will form a cuspidal curve on the parallel surface. Roberts determines the order of the cuspidal curve as double that of the surface of centres, and confirms his result by observing, that in the limiting case $k=\infty$, the locus of points on the surface of centres for which a principal radius of curvature $=k$, is the section of the surface of centres by the plane infinity, counted twice, since k may be $\pm\infty$. The singularities of the parallel surface here assigned are sufficient to determine the remainder by the help of the general theory of reciprocal surfaces hereafter to be explained.

In the case of the parallel to a quadric, it appears from what has been stated, that the reciprocal is of the fourth degree, and having no cuspidal curve, but having a nodal conic. The parallel itself is of the twelfth degree; its cuspidal curve is of the twenty-fourth order, being the complete intersection of a quartic with a sextic surface. The nodal curve is of the twenty-sixth order, and includes five conics, one in each of the principal planes, and two in the plane infinity, namely,

the section of the quadric itself and the circle at infinity. The remainder of the nodal curve consists of 16 right lines, each meeting the circle at infinity.*

PEDALS.

514. The locus of the feet of perpendiculars let fall from any fixed point on the tangent planes of a surface, is a derived surface to which French mathematicians have given a distinctive name, "podaire," which we shall translate as the *pedal* of the given surface. From the pedal may, in like manner, be derived a new surface, and from this another, &c., forming a series of second, third, &c., pedals. Again, the envelope of planes drawn perpendicular to the radii vectores of a surface, at their extremities, is a surface of which the given surface is a pedal, and which we may call the first negative pedal. The surface derived in like manner from this is the second negative pedal, and so on. Pedal curves and surfaces have been studied in particular by W. Roberts, *Liouville*, vols. X. and XII., by Tortolini, and by Hirst, Tortolini's *Annali*, vol. II. p. 95; see also the corresponding theory for plane curves, *Higher Plane Curves*, Art. 121. We shall here give some of their results, but must omit the greater part of them which relate to problems concerning rectification, quadrature, &c., and do not enter into the plan of this treatise. If Q be the foot of the perpendicular from O on the tangent plane at any point P, it is easy to see that the sphere described on the diameter OP touches the locus of Q; and consequently the normal at any point Q of the pedal passes through the middle point of the corresponding radius vector OP. It immediately follows hence, that the perpendicular OR on the tangent plane at Q lies in the plane POQ, and makes the angle $QOR = POQ$, so that the right-angled triangle QOR is similar to POQ; and if we call the angle QOR, α, so that the first perpendicular OQ is connected

* The parallel to a curve in space might also have been discussed. This is a tubular surface (Art. 446).

with the radius vector by the equation $p = \rho \cos \alpha$, then the second perpendicular OR will be $\rho \cos^2\alpha$, and so on.*

It is obvious that if we form the polar reciprocals of a curve or surface A and of its pedal B, we shall have a curve or surface a which will be the pedal of b; hence, if we take a surface S and its successive pedals S_1, S_2, $\ldots S_n$, the reciprocals will be a series S', S'_{-1}, S'_{-2}, $\ldots S'_{-n}$, those derived in the latter case being negative pedals.

It is also obvious that the first pedal is the *inverse* of the polar reciprocal of the given surface (that is to say, the surface derived from it by substituting in its equation, for the radius vector, its reciprocal); and that the inverse of the series S_1, S_2, $\ldots S_n$ will be the series S', S'_{-1}, $\ldots S'_{-n-1}$.

INVERSE SURFACES.

515. As we may not have the opportunity to return to the general theory of inversion, we give in this place the following statement (taken from Hirst, *Tortolini*, vol. II. p. 165) of the principal properties of inverse surfaces (see *Higher Plane Curves*, Arts. 122, 281).

(1) Three pairs of corresponding points on two inverse surfaces lie on the same sphere (and two pairs of corresponding points on the same circle) which cuts orthogonally the unit sphere whose centre is the origin.

(2) By the property of a quadrilateral inscribed in a circle the line ab joining any two points on one curve makes the same angle with the radius vector Oa, that the line joining the corresponding points $a'b'$ makes with the radius vector Ob'. In the limit then, if ab be the tangent at any point a, the corresponding tangent on the inverse curve makes the same angle with the radius vector.

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* Thus the radius vector to the n^{th} pedal is of length $\rho \cos^n \alpha$, and makes with the radius vector to the curve the angle $n\alpha$. Using this definition of the method of derivation, Roberts has considered fractional derived curves and surfaces. Thus for $n = \frac{1}{2}$, the curve derived from the ellipse is Cassini's oval. An analogous surface may be derived from the ellipsoid.

(3) In like manner for surfaces, two corresponding tangent planes are equally inclined to the radius vector, the two corresponding normals lying in the same plane with the radius vector, and forming with it an isosceles triangle whose base is the intercepted portion of the radius vector.

(4) It follows immediately from (2), that the angle which two curves make with each other at any point is equal to that which the inverse curves make at the corresponding point.

(5) In like manner it follows from (3), that the angle which two surfaces make with each other at any point is equal to that which the inverse surfaces make at the corresponding point.

(6) The inverse of a line or plane is a circle or sphere passing through the origin.

(7) Any circle may be considered as the intersection of a plane, and a sphere A through the origin. Its inverse, therefore, is another circle, which is a section of the cone whose vertex is the origin, and which stands on the given circle.

(8) The centre of the second circle lies on the line joining the origin to a, the vertex of the cone circumscribing the sphere A along the given circle. For a is evidently the centre of the sphere B which cuts A orthogonally. The plane, therefore, which is the inverse of A cuts B' the inverse of B orthogonally, that is to say, in a great circle, whose centre is the same as the centre of B'. But the centres of B and of B' lie in a right line through the origin.

(9) To a circle osculating any curve, evidently corresponds a circle osculating the inverse curve.

(10) For inverse surfaces, the centres of curvature of two corresponding normal sections lie in a right line with the origin. To the normal section a at any point m corresponds a curve a' situated on a sphere A passing through the origin; and the osculating circle c' of a' is the inverse of c the osculating circle of a. If now a_1 be the normal section which

touches a' at the point m', then, by Meunier's theorem, the centre of c' is the projection on its plane of the centre of c_1 the osculating circle of a_1. But the normal $m'c_1$ evidently touches the sphere A at m' so that c_1 is the vertex of the cone circumscribed to A along c', and theorem (10) therefore follows from theorem (8).

(11) To the two normal sections at m whose centres of curvature occupy extreme positions on the normal at m, will evidently correspond two sections enjoying the same property; therefore to *the two principal sections on one surface correspond two principal sections on the other, and to a line of curvature on one, a line of curvature on the other.**

In the case where the surface has no special relation to the plane or circle at infinity it is easy to see, as at *Higher Plane Curves*, p. 106, that the inverse of a surface is of the degree $2m$, and class $3m + 2a + n = m^2 + 2m$, that it passes m times through the origin and m times through the circle at infinity; and hence that the degree and class of the first pedal are

$$2n,\ m + 2a + 3n,$$

and of the first negative pedal $3m + 2a + n$ and $2m$.

[515*a*. If a system of curves satisfy the equation

$$ldx + mdy + ndz = 0$$

the inverse system (the origin being the centre of inversion and the radius of inversion unity) will satisfy the equation

$$l_1dx_1 + m_1dy_1 + n_1dz_1 = 0$$

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* Hart's method of obtaining focal properties by inversion (explained *Higher Plane Curves*, Art. 281) is equally applicable to curves in space and to surfaces. The inverse of any plane curve is a curve on the surface of a sphere, and in particular the inverse of a plane conic is the intersection of a sphere with a quadric cone. And as shown (*Higher Plane Curves*, Art. 281) from the focal property of the conic $\rho + \rho' =$ const. is inferred a focal property of the curve in space $l\rho + m\rho' + n\rho'' = 0$. So, in like manner, the inverse of a bicircular quartic is a curve in space with similar focal properties. (See Casey on Cyclides and Sphero-Quartics, *Phil. Trans.*, vol. 161; Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques*, Paris, 1873.) A surface which is its own inverse with regard to any point has been called an *anallagmatic* surface.

where

$$l_1 = l - \frac{2x}{r^2}P$$

with similar values of m_1 and n_1 where

$r^2 = x^2 + y^2 + z^2$, $P = lx + my + n^2$; also $l_1^2 + m_1^2 + n_1^2 = 1$.

The torsion of a curve whose normal is l, m, n at all points is (Art. 368a) given by

$$-\frac{ds^2}{\tau} = \begin{vmatrix} dx & dy & dz \\ l & m & n \\ dl & dm & dn \end{vmatrix}.$$

And the corresponding torsion $\left(\frac{1}{\tau_1}\right)$ (of the curve whose normals are l_1, m_1, n_1) for the inverse direction dx_1, dy_1, dz_1 is given by a like formula. By using these expressions we can prove

$$-\frac{1}{\tau_1} = \frac{1}{r^2\tau}\left(\frac{ds}{ds_1}\right)^2 = \frac{r^2}{\tau}.$$

Now $\frac{ds_1}{ds} = \frac{r_1}{r} = \frac{1}{r^2}$ and may be described as the linear magnification. Thus the new torsion with changed sign is equal to the original torsion divided by the linear magnification.

In particular if U = constant represents a surface (or a family of surfaces) the equation $dU = 0$ gives an equation of the form

$$ldx + mdy + ndz = 0$$

and the curves whose normals are l, m, n are geodesics. Thus *if a surface be inverted from any point the new geodesic torsion for any given direction is equal with changed sign to the old geodesic torsion of the corresponding direction divided by the linear magnification.* Since geodesic torsion vanishes for a line of curvature we see that (11) of Art. 515 is a special case of this theorem.]

516. The first pedal of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, being the inverse of the reciprocal ellipsoid, has for its equation

$$a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2.$$

This surface is Fresnel's "Surface of Elasticity". The inverse of a system of confocals cutting at right angles is evidently a system of surfaces of elasticity cutting at right angles; the lines of curvature therefore of the surface of elasticity are determined as the intersection with it of two surfaces of the same nature derived from concyclic quadrics.

The origin is evidently a nodal point on this surface, and the imaginary circle in which any sphere cuts the plane at infinity is a nodal line on the surface.

NEGATIVE PEDALS.

517. Cayley first obtained the equation of the first *negative pedal* of a quadric, that is to say, of the envelope of planes drawn perpendicular to the central radii at their extremities. It is evident that if we describe a sphere passing through the centre of the given quadric, and touching it at any point $x'y'z'$, then the point xyz on the derived surface which corresponds to $x'y'z'$ is the extremity of the diameter of this sphere, which passes through the centre of the quadric. We thus easily find the expressions

$$x = x'\left(2 - \frac{t}{a^2}\right), \quad y = y'\left(2 - \frac{t}{b^2}\right), \quad z = z'\left(2 - \frac{t}{c^2}\right),$$

where $$t = x'^2 + y'^2 + z'^2.$$

Solving these equations for x', y', z' and substituting their values in the two equations

$$xx' + yy' + zz' = x'^2 + y'^2 + z'^2, \qquad \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} = 1,$$

we get $$\frac{x^2}{\left(2 - \frac{t}{a^2}\right)} + \frac{y^2}{\left(2 - \frac{t}{b^2}\right)} + \frac{z^2}{\left(2 - \frac{t}{c^2}\right)} = t,$$

$$\frac{x^2}{a^2\left(2 - \frac{t}{a^2}\right)^2} + \frac{y^2}{b^2\left(2 - \frac{t}{b^2}\right)^2} + \frac{z^2}{c^2\left(2 - \frac{t}{c^2}\right)^2} = 1.$$

Now the second of these equations is the differential, with respect to t, of the first equation; and the required surface is therefore represented by the discriminant of that equation,

which we can easily form, the equation being only of the fourth degree. If we write this biquadratic

$$At^4 + 4Bt^3 + 6Ct^2 + 4Dt + E,$$

it will be found that A and B do not contain x, y, z, while C, D, E contain them, each in the second degree. Now the discriminant is of the sixth degree in the coefficients, and is of the form $A\phi + B^2\psi$; consequently it can contain x, y, z only in the tenth degree. This therefore is the degree of the surface required.

It appears, as in other similar cases, that the section by one of the principal planes z consists of the discriminant of

$$\frac{x^2}{2 - \frac{t}{a^2}} + \frac{y^2}{2 - \frac{t}{b^2}} = t,$$

which is a curve of the sixth degree, and is the first negative pedal of the corresponding principal section of the ellipsoid, together with the conic, counted twice, obtained by writing $t = 2c^2$, in the last equation. This conic, which is a double curve on the surface, touches the curve of the sixth degree in four points. The double points on the principal planes evidently answer to points on the ellipsoid, for which

$$t = x'^2 + y'^2 + z'^2 = 2a^2 \text{ or } 2b^2 \text{ or } 2c^2.$$

There is a cuspidal conic at infinity, and, besides, a finite cuspidal curve of the sixteenth degree.

CHAPTER XV.

SURFACES OF THE THIRD DEGREE.

519. THE general theory of surfaces, explained in Chap. XI., gives the following results, when applied to cubic surfaces. The tangent cone whose vertex is any point, and which envelopes such a surface, is, in general, of the sixth degree, having six cuspidal edges and no ordinary double edge. It is consequently of the twelfth class, having twenty-four stationary, and twenty-seven double tangent planes. Since then through any line twelve tangent planes can be drawn to the surface, any line meets the reciprocal in twelve points; and the reciprocal is, in general, of the twelfth degree. Its equation can be found as at *Higher Plane Curves*, Art. 91. The problem is the same as that of finding the condition that the plane

$$\alpha x + \beta y + \gamma z + \delta w = 0$$

should touch the surface. Multiply the equation of the surface by δ^3, and then eliminate δw by the help of the equation of the plane. The result is a homogeneous cubic in x, y, z, containing also $\alpha, \beta, \gamma, \delta$ in the third degree. The discriminant of this equation is of the twelfth degree in its coefficients, and therefore of the thirty-sixth in $\alpha, \beta, \gamma, \delta$; but this consists of the equation of the reciprocal surface multiplied by the irrelevant factor δ^{24}. The form of the discriminant of a homogeneous cubic function in x, y, z is $64S^3 + T^2$ (*Higher Plane Curves*, Art. 224). The same, then, will be the form of the reciprocal of a surface of the third degree, S being of the fourth, and T of the sixth degree in $\alpha, \beta, \gamma, \delta$; (that is to say, S and T are *contravariants* of the given equation of the above degrees). It is easy to see that they are also of the same degrees in the coefficients of the given equation.

520. Surfaces may have either multiple points or multiple lines. When a surface has a double line of the degree p, then any plane meets the surface in a section having p double points. There is, therefore, the same limit to the degree of the double curve on a surface of the n^{th} degree that there is to the number of double points on a curve of the n^{th} degree. Since a curve of the third degree can have only one double point, *if a surface of the third degree has a double line, that line must be a right line.** A cubic having a double line is necessarily a ruled surface, for every plane passing through this line meets the surface in the double line, reckoned twice, and in another line; but these other lines form a system of generators resting on the double line as director. If we make the double line the axis of z, the equation of the surface will be of the form

$$(ax^3 + 3bx^2y + 3cxy^2 + dy^3) + z\,(a'x^2 + 2b'xy + c'y^2) + (a''x^2 + 2b''xy + c''y^2) = 0,$$

which we may write $u_3 + zu_2 + v_2 = 0$. At any point on the double line there will be a pair of tangent planes $z'u_2 + v_2 = 0$. But as z' varies this denotes a system of planes in involution (*Conics*, Art. 342). Hence *the tangent planes at any point on the double line are two conjugate planes of a system in involution.*

There are two values of z', real or imaginary, which will make $z'u_2 + v_2$ a perfect square; there are, therefore, two points on the double line at which the tangent planes coincide; and any plane through either of them meets the surface in a section having this point for a cusp. If the values of these squares be X^2 and Y^2, it is evident that u_2 and v_2 can each be expressed in the form $lX^2 + mY^2$. If, then, we turn round the axes so

* If a surface have a double or other multiple line, the reciprocal formed by the method of the last article would vanish identically; because then *every* plane meets the surface in a curve having a double point, and, therefore, the plane $\alpha x + \beta y + \gamma z + \delta w$ is to be considered as touching the surface, independently of any relation between $\alpha, \beta, \gamma, \delta$. The reciprocal can be found in this case by eliminating x, y, z, w between $u = 0$, $\alpha = u_1$, $\beta = u_2$, $\gamma = u_3$, $\delta = u_4$.

as to have for coordinate planes the planes X, Y, that is to say, the tangent planes at the cuspidal points, then every term in the equation will be divisible by either x^2 or y^2, and the equation may be reduced to the form $zx^2 = wy^2$.*

In this form it is evident that the surface is generated by lines $y = \lambda x$, $z = \lambda^2 w$, intersecting the two directing lines xy, zw; and the generators join the points of a system on zw to the pairs of corresponding points of a system in involution on xy, homographic with the first system. Any plane through zw meets the surface in a pair of right lines, and is to be regarded as touching the surface in the two points where these lines meet zw. Thus, then, as the line xy is a line, every point of which is a double point, so the line zw is a line, every plane through which is a double tangent plane. The reciprocal of this surface, which is that considered in Art. 468, is of like nature with itself.

The tangent cone whose vertex is any point, and which envelopes the surface, consists of the plane joining the point to the double line, reckoned twice, and a proper tangent cone of the fourth degree. When the point is on the surface the cone reduces to the second degree.

521. There is one case, to which my attention was called by Prof. Cayley, in which the reduction to the form $zx^2 = wy^2$ is not possible. If u_2 and v_2, in the last article, have a common factor, then choosing the plane represented by this for one

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* It is here supposed that the planes X, Y, the double planes of the system in involution, are real. We can always, however, reduce to the form $w(x^2 \pm y^2) + 2zxy$, the upper sign corresponding to real, and the lower to imaginary, double planes, for

$$(z - w)\{(x + y)^2 + (x - y)^2\} + 2(z + w)(x + y)(x - y) \equiv 4(x^2 z - wy^2).$$

In the latter case the double line is altogether "really" in the surface, every plane meeting the surface is a section having the point where it meets the line for a real node. In the former case this is only true for a limited portion of the double line, sections which meet it elsewhere having the point of meeting for a conjugate point, the two cuspidal points marking these limits on the double line. A right line, every point of which is a cusp, cannot exist on a cubic unless when the surface is a cone.

of the coordinate planes, and $cx+dy$ for another, we can easily throw the equation of the surface into the form

$$y^3+x\,(zx+wy)=0.$$

The plane x touches the surface along the whole length of the double line, and meets the surface in three coincident right lines. The other tangent plane at any point coincides with the tangent plane to the hyperboloid $zx+wy$. This case may be considered as a limiting case of that considered in the last article; viz., when the double director xy coincides with the single one wz. The following generation of the surface may be given: Take a series of points on xy, and a homographic series of planes through it, then the generator of the cubic through any point on the line lies in the corresponding plane, and may be completely determined by taking as director a plane cubic having a double point where its plane meets the double line, and such that one of the tangents at the double point lies in the plane which corresponds to the double point considered as a point in the double line.* [The reciprocal of this surface is also of like nature with itself.]

522. The argument which proves that a proper cubic curve cannot have more than one double point does not apply to surfaces. In fact, the line joining two double points, since it is to be regarded as meeting the surface in four points, must lie altogether in the surface; but this does not imply that the surface breaks up into others of lower dimensions. The consideration of the tangent cone, however, supplies a limit to the number of double points on the surface. We have seen (Art. 279) that the tangent cone is of the sixth degree, and has six cuspidal edges, and it is known that a curve of the sixth degree having six cusps can have only four other double points. Since, then, every double point on the surface adds a double edge to the tangent cone, a cubical surface can at most have four double points.

* The reader is referred to an interesting geometrical memoir on cubical ruled surfaces by Cremona, *Atte del Reale Instituto Lombardo*, vol. II. p. 291.

It is necessary to distinguish the various kinds of node which the surface may possess. (*A*) At an ordinary node* (Art. 283) the tangent plane is replaced by a quadric cone. The line joining the node to any assumed point, is, as has been said, a double edge of the tangent cone from the latter point; and since to the tangent cone from any point corresponds a plane section of the reciprocal surface, this double edge evidently reduces by two the degree of the reciprocal, or the class of the given surface. (*B*) The quadric cone may degenerate into a pair of planes. Such a node may be called a *binode;* the planes the *biplanes*, and their intersection the *edge*. In the case first considered, it is easy to see that the tangent planes to any tangent cone along its double edge are the planes drawn through this line to touch the nodal cone. When, therefore, the nodal cone reduces to two planes, these tangent planes coincide, and the line to the binode is a cuspidal edge of the tangent cone. A binode, therefore, ordinarily reduces the class of the surface by three. A cubic cannot have more than three binodes, since a proper sextic cone cannot have more than nine cuspidal edges. But there may be special cases of binodes. (1) At an ordinary binode B_3 the edge does not lie on the surface; but if it does,† the binode is special B_4, and reduces the class of the surface by *four*. Thus, let xyz be the binode, x, y the biplanes, the general equation of the surface will be of the form $u_3 + xy = 0$, where $u_3 = c_0z^3 + 3c_1z^2x + 3c_2z^2y + \&c.$ The case where $c_0 = 0$ is the special one under consideration. This kind of binode may be considered as resulting from the union of two conical nodes. (2) In the special case last considered, the surface is touched along the edge by a plane $c_1x + c_2y$, which commonly is distinct from one of the biplanes; but it may

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* Cayley calls the kind of node here considered a *cnic-node*, and it is referred to accordingly as C_2. [It is now usually called a conic node.]

† [For the general surface the distinction between B_3 and B_4 is that in the former case the six lines of closest contact $u_2 = u_3 = 0$ are distinct from the edge, but in the latter two coincide with this edge. Thus B_4 and all higher binodes are of a specialised nature on the cubic.]

coincide with one of them, that is to say, we may have either c_1 or $c_2 = 0$. In this case, the binode B_5 reduces the class of the surface by *five*. Such a point may be considered as resulting from the union of a conical node and binode. (3) Lastly, we may have x a factor in all the terms of u_3 except y^3, and we have then a binode B_6, which may be regarded as resulting from the union of three conical nodes, and which reduces the class of the surface by *six*. In this case the edge is said to be *oscular*.* (*C*) The two biplanes may coincide, when we have what may be called a *unode* U_6, which reduces the class of the surface by *six;* the equation then being reducible to the form $u_3 + x^2 = 0$. The uniplane x meets the surface in three right lines, which are commonly distinct; but either, two of these may coincide, or all three may coincide, when we have special cases of unodes, U_7, U_8 which reduce the class of the surface by seven and eight respectively. U_6 may be regarded as equivalent to three conical nodes, U_7 to two conical and a binode, U_8 to two binodes and a conical.

[522*a*. The reduction in class effected by these various kinds of nodes can be seen by inquiring how many tangent planes can be drawn through the line zw, distinct from the plane z. Thus for the case B_5 the points of contact are the intersections of the three surfaces

$$xyw + kz^2x + zu + v = 0$$
$$yw + kz^2 + zu' + v' = 0$$
$$xw + zu'' + v'' = 0$$

where u and v are respectively a binary quadratic and cubic in x and y, and the accents denote differentiation. Thus they are the intersections of the cubic with the cubic cones

* In general, if a surface is touched along a right line by a plane, the right line counts twice as part of the complete intersection of the surface by the plane, the remaining intersection being of the degree $n - 2$. The line may, however, count three times, the remaining intersection being only of the degree $n - 3$. Cayley calls the line *torsal* in the first case, *oscular* in the second. He calls it *scrolar* if the surface merely contain the right line, in which case there is ordinarily a different tangent plane at each point of the line.

$$z(u - xu') + v - xv' = 0$$
$$kz^2x + z(u - yu'') + v - yv'' = 0.$$

These cones have seven common edges other than xy which is a double edge on the first, and hence the class of the cubic surface is seven. For a general discussion applicable to any surface see Schläfli (*Phil. Trans.*, 1863).

522*b*. The equivalence of higher nodes to certain combinations of C_2 and B_3 can been seen by Segre's method (522*c*) or in an elementary manner, thus

$$xy + z(z - k)(c_1x + c_2y) + z(ax^2 + 2hxy + by^2) + u_3 = 0$$

represents a cubic having conic nodes at $(0, 0, 0)$ and $(0, 0, k)$. Let k diminish to zero and we get a cubic with a B_4 point. Similarly if c_1 be put zero we see that $B_5 = C_2 + B_3$. If c_1 and a be both zero, there are binodes at both points and coalescence gives rise to B_6.

This result is apparently in conflict with the statement $B_6 = 3C_2$, but the difficulty is explained by considering the tangent cone from any external point. On it we have two cuspidal edges uniting into an oscnodal edge, but an oscnode in the theory of plane curves has to be considered as equivalent to three nodes, not two cusps (see Basset's *Surfaces*,* p. 134), if Plücker's formulæ are to apply. And in this sense $B_6 = 3C_2$ and not $2B_3$. It will also be shown in Ex. 3 below that three conic nodes may coalesce to form either B_6 or U_6.

522*c*. Segre (*Ann. di Mat.* (2) xxv.) has developed a powerful method of decomposing complex multiple points of any order into simpler constituents. He shows that an s-ple point at O may be replaced, as regards the determination of the intersections of the surface with any other surface or curve through O, by the following collection of fictitious points:—

(*a*) An s-ple point at O;

* *A Treatise on the Geometry of Surfaces*, by A. B. Basset (Cambridge, 1910).

(*b*) Points O_1, O_2 . . . of order s_1, s_2, . . . at points consecutive to O along certain singular generators of the nodal cone at O ($s_r \overline{\overline{<}} s$);

(*c*) Points O_{p1}, O_{p2}, . . . of order s_{p1}, s_{p2} . . . at points consecutive to O^p in certain specified directions ($s_{pr} \overline{\overline{<}} s_p$); and so on.

To determine in how many points a curve meets the surface at O we have then only to count with their proper multiplicity the number of these fictitious points that lie on it.

The following very brief sketch of the proof of these statements is sufficient for the purpose of application to the cubic surface; the full proof depends on the principles of Birational Transformation mentioned in the last chapter of this book.

Let ω be a quadric function of x, y, z and Ω the corresponding function of X, Y, Z, and let Q denote the conic $\Omega = W = 0$.

The surface

$$\phi \equiv u_s w^{n-s} + u_{s+1} w^{n-s-1} + \ . \ . \ . \ + u_n = 0$$

having an s-ple point at O is transformed into

$$\Phi \equiv U_s \Omega^{n-s} + U_{s+1} \Omega^{n-s-1} W + \ . \ . \ . \ U_n W^n = 0$$

by the birational transformation

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{w}{\omega} = \frac{\Omega}{W}.$$

Points on the plane W correspond to the various elements of direction round O, and any curve through O will meet ϕ in $s+k$ points at O if the point on W corresponding to the direction of its tangent is a k-ple point on Φ.

For instance, if u_s has a nodal generator (γ) which is also a generator of u_{s+1}, Φ will have a conic node at the point on W corresponding to γ, and any curve touching γ at O will meet ϕ in $s+2$ points. Should we in this way obtain on Φ a special point O_p of any kind, its analysis is continued by another transformation, but it is to be observed that singular points arising from special relations of Ω to the U functions are not considered, for Ω is completely at our disposal and

such points may be prevented. Of course the conic Q itself, a singular curve of order $n - s$, is irrelevant.

Applying now the foregoing theory to the cubic surface it transforms into

$$XY\Omega + W\{c_0Z^3 + (c_1X + c_2Y)Z^2 + U_2Z + U_3\} = 0.$$

If c_0 be zero XYW is a conic node, and if in addition c_1 vanishes this point is a binode.

This gives us $B_4 = 2C_2$ and $B_5 = C_2 + B_3$. The further condition that the edge of the binode may lie on the cone obtained by equating to zero the coefficient of Z, that is, that the binode on Φ may be B_4, will be found to be that u_2 contains y as a factor or that ϕ has a B_6 point. Hence

$$B_6 = C_2 + B_4 = 3C_2.$$

Unodes may be analysed in a similar way.]

523. Distinguishing cubic surfaces according to the singularities described in the preceding articles, we can enumerate twenty-three possible forms of cubics, which are exhibited in the following table. [The last column gives the number of distinct lines on the surface.]

No.	Nodes.	Class.	[Lines.]
1	o	12	27
2	C_2	10	21
3	B_3	9	15
4	$2C_2$	8	16
5	B_4	8	10
6	$B_3 + C_2$	7	11
7	B_5	7	6
8	$3C_2$	6	12
9	$2B_3$	6	7
10	$B_4 + C_2$	6	7
11	B_6	6	3
12	U_6	6	6
13	$B_3 + 2C_2$	5	8
14	$B_5 + C_2$	5	4
15	U_7	5	3
16	$4C_2$	4	9
17	$2B_3 + C_2$	4	5
18	$B_4 + 2C_2$	4	5
19	$B_6 + C_2$	4	2
20	U_8	4	1
21	$3B_3$	3	3

The number twenty-three is completed by the two kinds of ruled surfaces or scrolls described in Arts. 520, 521, each of which is of the third class.*

Ex. 1. What is the degree of the reciprocal of $xyz = w^3$?

Ans. There are three biplanar points in the plane w, and the reciprocal is a cubic.

Ex. 2. What is the reciprocal of $\frac{l}{x} + \frac{m}{y} + \frac{n}{z} + \frac{p}{w} = 0$?

Ans. This represents a cubic having the vertices of the pyramid $xyzw$ for double points; and the reciprocal must be of the fourth degree.

The equation of the tangent plane at any point $x'y'z'w'$ can be thrown into the form $\frac{lx}{x'^2} + \frac{my}{y'^2} + \frac{nz}{z'^2} + \frac{pw}{w'^2} = 0$, whence it follows that the condition that

$$\alpha x + \beta y + \gamma z + \delta w = 0$$

should be a tangent plane is

$$(l\alpha)^{\frac{1}{2}} + (m\beta)^{\frac{1}{2}} + (n\gamma)^{\frac{1}{2}} + (p\delta)^{\frac{1}{2}} = 0,$$

an equation which, cleared of radicals, is of the fourth degree.† Generally the reciprocal of $ax^n + by^n + cz^n + dw^n$ is of the form

$$A\alpha^{\frac{n}{n-1}} + B\beta^{\frac{n}{n-1}} + C\gamma^{\frac{n}{n-1}} + D\delta^{\frac{n}{n-1}} = 0.$$

(*Higher Plane Curves*, p. 73.)

The tangent cone to this surface, whose vertex is any point on the surface,

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* The effect of the nodes C_2, B_3, U_6 on the class of the surface was pointed out by me, *Cambridge and Dublin Mathematical Journal*, 1847, vol. II. p. 65; and the twenty-seven right lines on the surface were accounted for in each case where we have any combination of these nodes, *Cambridge and Dublin Mathematical Journal*, 1849, vol. IV. p. 252. The special cases B_4, B_5, B_6, U_7, U_8 were remarked by Schläfli, *Phil. Trans.*, 1863, p. 201. See also Cayley's Memoir on Cubic Surfaces, *Phil. Trans.*, 1869, pp. 231-326. [The standard forms to which the equations of these various kinds of cubics can be reduced and the equations of the lines on them are given in *The Twenty-seven Lines upon the Cubic Surface*, by A. Henderson (Cambridge, 1911). This work, Blythe's *Models of Cubic Surfaces* (Cambridge, 1905) and Klein's Memoir (*Math. Ann.* VI.) should be consulted to obtain an idea of the shapes of these surfaces.]

† Writing x, y, z, w in place of $l\alpha$, $m\beta$, $n\gamma$, $p\delta$ respectively, the equation of the reciprocal surface is

$$\sqrt{(x)} + \sqrt{(y)} + \sqrt{(z)} + \sqrt{(w)} = 0,$$

which rationalised is

$$(x^2 + y^2 + z^2 + w^2 - 2yz - 2zx - 2xy - 2xw - 2yw - 2zw)^2 - 64xyzw = 0,$$

the surface commonly known as Steiner's quartic. It has three double lines meeting in a point; every tangent plane cuts it in two conics. [See Art. 554*a*.]

being of the fourth degree, and having four double edges, must break up into two cones of the second degree.

A cubic having four double points is also the envelope of

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2l\beta\gamma + 2m\gamma\alpha + 2n\alpha\beta,$$

where a, b, c, l, m, n represent planes; and $\alpha:\gamma$, $\beta:\gamma$ are two variable parameters. It is obvious that the envelope is of the third degree; and it is of the fourth class; since if we substitute the coordinates of two points we can determine four planes of the system passing through the line joining these points.

Generally the envelope of $a\alpha^n + b\beta^n + \&c.$ is of the degree $3\,(n-1)^2$ and of the class n^2. The tangent cone from any point is of the degree $3n\,(n-1)$. It has a cuspidal curve whose order is the same as the order of the condition that $U + \lambda V$ may represent a plane curve having a cusp, U and V denoting plane curves of the n^{th} degree; or, in other words, is equal to the number of curves of the form $U + \lambda V + \mu W$ which can have a cusp. The surface has a nodal curve whose order is the same as the number of curves of the form $U + \lambda V + \mu W$ which can have two double points. For these numbers, see *Higher Algebra*, Lesson XVIII.

[Ex. 3. Show that three conic nodes coalesce into a B_6 or a U_6 point according as the ultimate directions of the lines joining them are coincident or distinct.

The most general cubic with three conic nodes is

$$pw^3 + (ax + by + cz)w^2 + (fyz + gzx + hxy)w + kxyz = 0.$$

Write in this for x, y, and z, $ty + z$, $2ty + z$, and $2t^2x + 3ty + z$ respectively, so that the three nodes are at the points o, t, $2t$ of the plane conic

$$x:y:z:w::1:-t:t^2:0.$$

If the nodes be made to coincide by making t approach zero we simply get three planes through zw unless some of the constants are made to become indefinitely great at the same time. We will get a surface with a B_6 point if

$$a = -\tfrac{1}{2}b = c = At^{-2}$$

and

$$f = -\tfrac{1}{2}g = h = Ft^{-2}.$$

For the equation becomes in the limit

$$pw^3 + 2Axw^2 + 2Fy^2w - 2Fxzw + kz^3 = 0$$

and yzw is a B_6 point.

If, however, we write $y + z + tx$ for x in the general 3-nodal form and make t zero we get the case where the lines joining the nodes remain distinct. Putting

$$a = -b = -c = At^{-1}$$

we get

$$pw^3 + Axw^2 + w(\overline{f+g+h}\,yz + gz^2 + hy^2) + k(y^2z + yz^2) = 0$$

on which yzw is a U_6 point.

Ex. 4. Cubic surfaces of the third class cannot have conic nodes. For a cone of the third class cannot have a nodal edge. This is why combinations like $2C_2 + B_5$ are impossible.

Ex. 5. $2B_4$ is inadmissible because a quartic cone cannot have two tacnodal edges.

Ex. 6. A cubic scroll can be generated by a right line meeting two conics with one common point and a line meeting both.

Ex. 7. Three tangent planes to a cubic scroll meet it in a circle, namely, the real tangent planes, one through each of the real lines joining intersections of the scroll with the imaginary circle at infinity.

Ex. 8. The polar quadric with respect to a cubic scroll of a point on a cuspidal tangent plane is a cone, and conversely.]

CANONICAL FORM.—THE HESSIAN.

524. The equation of a cubic having no multiple point may be thrown into the form $ax^3 + by^3 + cz^3 + dv^3 + ew^3 = 0$, where x, y, z, v, w represent planes, and where for simplicity we suppose that the constants implicitly involved in x, y, &c., have been so chosen, that the identical relation connecting the equations of any five planes (Art. 38) may be written in the form $x + y + z + v + w = 0$. In fact, the general equation of the third degree contains twenty terms, and therefore nineteen independent constants, but the form just written contains five terms, and, therefore, four expressed independent constants, while, besides, the equation of each of the five planes implicitly involves three constants. The form just written, therefore, contains the same number of constants as the general equation. This form given by Sylvester in 1851 (*Cambridge and Dublin Mathematical Journal*, vol. VI., p. 199) is very convenient for the investigation of the properties of cubic surfaces in general.*

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* It was observed (*Higher Plane Curves*, Art. 25) that two forms may apparently contain the same number of independent constants, and yet that one may be less general than the other. Thus, when a form is found to contain the same number of constants as the general equation, it is not absolutely demonstrated that the general equation is reducible to this form; and Clebsch has noticed a remarkable exception in the case of curves of the fourth order (see note, Art. 235). In the present case, though Mr. Sylvester gave his theorem without further demonstration, he states that he was in possession of a proof that the general equation could be reduced to the sum of five cubes, and in but a single way. Such a proof has been published by Clebsch (*Crelle*, vol. LIX. p. 193). See also Gordan, *Math. Annalen*, v. 341; and on the general theory of cubic surfaces Cremona, *Crelle*, vol. LXVIII.; Sturm, *Synthetische Untersuchungen über Flächen dritter Ordnung*. Clebsch erroneously ascribes the theorem in the text to Steiner, who gave it in the year 1856 (*Crelle*, vol. LIII. p. 133); but this, as well as Steiner's other principal results, had been known in this country a few years before.

525. If we write the equation of the first polar of any point with regard to a surface of the n^{th} degree,

$$x'L + y'M + z'N + w'P = 0,$$

then, if it have a double point, that point will satisfy the equations

$$ax' + hy' + gz' + lw' = 0, \qquad hx' + by' + fz' + mw' = 0,$$
$$gx' + fy' + cz' + nw' = 0, \qquad lx' + my' + nz' + dw' = 0,$$

where a, b, &c., denote second differential coefficients corresponding to these letters, as we have used them in the general equation of the second degree. Now, if between the above equations we eliminate $x'y'z'w'$, we obtain the locus of all points which are double points on first polars. This is of the degree $4(n-2)$, and is, in fact, the *Hessian* (Art. 285). If we eliminate the $xyzw$ which occur in a, b, &c., since the four equations are each of the degree $(n-2)$, the resulting equation in $x'y'z'w'$ will be of the degree $4(n-2)^3$, and will represent the locus of points whose first polars have double points. Or, again, H is the locus of points whose polar quadrics are cones, while the second surface, which (see *Higher Plane Curves*, Art. 70) may be called the *Steinerian*, is the locus of the vertices of such cones. In the case of surfaces of the third degree, it is easy to see that the four equations above written are symmetrical between $xyzw$ and $x'y'z'w'$; and, therefore, that the Hessian and Steinerian are identical. Thus, then, *if the polar quadric of any point A with respect to a cubic be a cone whose vertex is B, the polar quadric of B is a cone whose vertex is A.* The points A and B are said to be corresponding points on the Hessian (see *Higher Plane Curves*, Art. 175, &c.).

526. *The tangent plane to the Hessian of a cubic at A is the polar plane of B with respect to the cubic.* For if we take any point A' consecutive to A and on the Hessian, then since the first polars of A and A' are consecutive and both cones, it appears (as at *Higher Plane Curves*, Art. 178) that their intersection passes indefinitely near B, the vertex of either cone; therefore the polar plane of B passes through

AA'; and, in like manner, it passes through every other point consecutive to A. It is, therefore, the tangent plane at A. And the polar plane of any point A on the Hessian of a surface of any degree is the tangent plane of the corresponding point B on the Steinerian. In particular, *the tangent planes to U along the parabolic curve are tangent planes to the Steinerian*; that is to say, in the case of a cubic *the developable circumscribing a cubic along the parabolic curve also circumscribes the Hessian.* If any line meet the Hessian in two corresponding points A, B, and in two other points C, D, the tangent planes at A, B intersect along the line joining the two points corresponding to C, D. [For if these be C', D', the polar quadrics of all points along AB have $ABC'D'$ as self-conjugate tetrahedron, and so $BC'D'$ is polar plane of A with respect to its polar quadric in particular.]

527. We shall also investigate the preceding theorems by means of the canonical form. The polar quadric of any point with regard to $ax^3+by^3+cz^3+dv^3+ew^3$ is got by substituting for w its value $-(x+y+z+v)$, when we can proceed according to the ordinary rules, the equation being then expressed in terms of four variables. We thus find for the polar quadric $ax'x^2+by'y^2+cz'z^2+dv'v^2+ew'w^2=0$. If we differentiate this equation with respect to x, remembering that $dw=-dx$, we get $ax'x=ew'w$; and since the vertex of the cone must satisfy the four differentials with respect to x, y, z, v, we find that the coordinates x', y', z', v', w' of any point A on the Hessian are connected with the coordinates x, y, z, v, w of B, the vertex of the corresponding cone, by the relations

$$ax'x=by'y=cz'z=dv'v=ew'w.$$

And since we are only concerned with mutual ratios of coordinates, we may take 1 for the common value of these quantities and write the coordinates of B, $\frac{1}{ax'}, \frac{1}{by'}, \frac{1}{cz'}, \frac{1}{dv'}, \frac{1}{ew'}$. Since the coordinates of B must satisfy the identical relation

$x + y + z + v + w = 0$, we thus get the equation of the Hessian

$$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} + \frac{1}{dv} + \frac{1}{ew} = 0,$$

or

$$bcdeyzvw + cdeazvwx + deabvwxy + eabcwxyz + abcdxyzv = 0.$$

This form of the equation shows that the line vw lies altogether in the Hessian, and that the point xyz is a double point on the Hessian; and since the five planes x, y, z, v, w give rise to ten combinations, whether taken by twos or by threes, we have Sylvester's theorem that *the five planes form a pentahedron whose ten vertices are double points on the Hessian and whose ten edges lie on the Hessian.* The polar quadric of the point xyz is $dv'v^2 + ew'w^2$, which resolves itself into two planes intersecting along vw, any point on which line may be regarded as the point B corresponding to xyz; thus, then, *there are ten points whose polar quadrics break up into pairs of planes; these points are double points on the Hessian, and the intersections of the corresponding pairs of planes are lines on the Hessian.* It is by proving these theorems independently * that the resolution of the given equation into the sum of five cubes can be completely established.

The equation of the tangent plane at any point of the Hessian may be written

$$\frac{x}{ax'^2} + \frac{y}{by'^2} + \frac{z}{cz'^2} + \frac{v}{dv'^2} + \frac{w}{ew'^2} = 0,$$

which, if we substitute for x', $\frac{1}{ax'}$, &c., becomes

$$ax'^2x + by'^2y + cz'^2z + dv'^2v + ew'^2w = 0,$$

but this is the polar plane of the corresponding point with regard to U.

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* It appears from *Higher Algebra*, Lesson XVIII., that a symmetric determinant of p rows and columns, each constituent of which is a function of the n^{th} order in the variables, represents a surface of the np^{th} degree having $\frac{1}{6}p\,(p^2 - 1)\,n^3$ double points; and thus that the Hessian of a surface of the n^{th} degree always has $10\,(n-2)^3$ double points.

[527*a*. Two recent papers prove the possibility and the uniqueness of the analytical reduction to this form in a comparatively simple manner. Baker* reduces the form (see Art. 533) $ace = bdf$ to $\sum_1^6 x^3_r = 0$, where $\sum_1^6 x_r \equiv 0$ and $\sum_1^6 h_r x_r \equiv 0$, and Bennett† reduces this last form to the sum of five cubes. The following geometrical proof given by way of illustration of the theorems of the last few articles, assumes the result from the *Higher Algebra* mentioned in the footnote to Art. 527, namely, that the ten quadrics represented by the first minors of the Hessian determinant have ten common points. The polar quadrics of these common points are therefore such that all the first minors of their discriminants vanish, or in other words, they are a pair of planes. These points are double points on the Hessian since their co-ordinates make $\frac{dH}{dx}$, &c., vanish. Let A be one of these ten points and let L be the line of intersection of the pair of planes forming its polar quadric. The polar quadrics of points on L are a family of cones $U = kV$ with vertices at A, and three members of this family break up into planes. Accordingly three of the ten before-mentioned points must lie on L and there are three corresponding lines passing through A. Thus the ten points lie by threes on lines which pass by threes through them, so there are ten lines like L. Now let the three lines through A be denoted by L_1, L_2, L_3, and let B_1, C_1; B_2, C_2; B_3, C_3 be respectively the other pair of points on each. Let X, Y, Z be the remaining three points. Consider the other two lines through B_1; one at least of the four points B_2, C_2, B_3, C_3 must lie on one of them, for there are not enough other points available; let it be B_2 and let Z be the third point on the line B_1B_2. Consider the section of the Hessian by the plane of L_1 and L_2. It consists of the three lines L_1, L_2, B_1B_2Z and of another. But this other line must pass through C_1, C_2, and Z since these three points are double points on the curve of section, hence

* Proc. *Lond. Math. Soc.*, vol. IX. † *Ibid.*, vol. X.

C_1, C_2, Z are collinear. In this way we see that the plane of any two lines contains a sixth point in addition to the five on the two lines. The points thus lie by sixes in planes of which three pass through each point, so there are five planes. If these be taken as planes of reference the equations of the Hessian and the cubic must be of the form given in Art. 527, and we see that the pentahedron of reference is unique.]

[527b. R. Russell* has discussed the focal surface of the congruence of lines joining corresponding points on the Hessian. This surface is the analogue of the Cayleyan in plane cubics. The following is an outline of his main results :—

Let P and P' be two corresponding points $\left(x \text{ and } \dfrac{1}{ax}\right)$ on the Hessian and let U and U' be the other two points $\left(x - \dfrac{\theta_1}{ax} \text{ and } x - \dfrac{\theta_2}{ax}\right)$ where PP' meets the Hessian.

The line PP' will touch the focal surface at points T and T' respectively harmonically conjugate to U and U' with respect to P and P', viz. the points $x + \dfrac{\theta_1}{ax}$ and $x + \dfrac{\theta_2}{ax}$, θ_1 and θ_2 being the roots of

$$\Sigma \frac{x}{ax^2 - \theta} = 0.$$

If V and V' are the correspondents to U and U', the planes PVV' and $P'VV'$ touch the Hessian at P and P' and the two tangent lines at P and at P' to the intersection of these planes with the Hessian give the directions in which we must proceed from P or P' to obtain the two consecutive lines of the congruence which meet PP'.

V and V' are each on the tangent plane to the Hessian at the other, and conversely two points U and U', which are the correspondents to two points having this relation, connect through two corresponding points P and P'.

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* *Proceedings of the Royal Irish Academy*, Ser. 3, vol. v. p. 462.

(These results can be easily verified by forming the co-ordinates of the various points and remembering that by virtue of the equations satisfied by x the equation for θ may be written

$$\Sigma \frac{Ax^4+Bx^2+C}{ax(ax^2-\theta)}=0 \quad A, B, \text{ and } C \text{ being arbitrary}).$$

Now V being given six points V' can be determined, since six tangents can be drawn to a uninodal plane quartic from the node; so that U being fixed there are six points U' such that the other two points of the Hessian lying on UU' are correspondents. These six lines UU' and the line UV make seven bitangents through U so that the *order of the congruence of bitangents is seven.* The class is three, for the lines joining the points $\alpha\beta$ and $\gamma\delta$, &c., in Ex. 3, Art. 529, are three bitangents lying in an arbitrary plane. The reader should consult the paper cited in the footnote for further details.]

528. If we consider all the points of a fixed plane, their polar planes envelope a surface, which (as at *Higher Plane Curves*, Art. 184) is also the locus of points whose polar quadrics touch the given plane. The parameters in the equation of the variable plane enter in the second degree; the problem is therefore that considered (Ex. 2, Art. 523) and the envelope is a cubic surface having four double points. The polar planes of the points of the section of the original cubic by the fixed plane are the tangent planes at those points, consequently this polar cubic of the given plane is inscribed in the developable formed by the tangent planes to the cubic along the section by the given plane (*Higher Plane Curves*, Art. 185). The polar plane of any point A of the section of the Hessian by the given plane touches the Hessian at the corresponding point B (Art. 526), and is, therefore, a common tangent plane of the Hessian and of the polar cubic now under consideration. But the polar quadric of B, being a cone whose vertex is A, is to be regarded as touching the given plane at A; hence B is also the point of contact of the polar plane of A with the polar cubic. We thus obtain a

theorem of Steiner's that *the polar cubic of any plane touches the Hessian along a certain curve.* This curve is the locus of the points B corresponding to the points of the section of the Hessian by the given plane. Now if points lie in any plane $lx+my+nz+pv+qw$, the corresponding points lie on the surface of the fourth degree $\frac{l}{ax}+\frac{m}{by}+\frac{n}{cz}+\frac{p}{dv}+\frac{q}{ew}$. Also the intersection of this surface with the Hessian is of the sixteenth degree, and includes the ten right lines xy, zw, &c. The remaining curve of the sixth degree is the curve along which the polar cubic of the given plane touches the Hessian. *The four double points lie on this curve; they are the points whose polar quadrics are cones touching the given plane.*

[The tangent cone to the polar cubic from any point A on it, being a cone of the fourth degree with four double edges, breaks up into the two quadric cones which are the envelopes respectively of the polar planes of points on the two lines in which the polar quadric of A meets the given plane. Now if A is a double point these two cones coincide and become the nodal cone; accordingly the two lines just mentioned must also coincide, which shows that the polar quadric of A becomes a cone touching the given plane along a line.]

529. If on the line joining any two points $x'y'z'$, $x''y''z''$, we take any point $x'+\lambda x''$, &c., it is easy to see that its polar plane is of the form $P_{11}+2\lambda P_{12}+\lambda^2 P_{22}$, where P_{11}, P_{22} are the polar planes of the two given points, and P_{12} is the polar plane of either point with regard to the polar quadric of the other. The envelope of this plane, considering λ variable, is evidently a quadric cone whose vertex is the intersection of the three planes. This cone is clearly a tangent cone to the polar cubic of any plane through the given line, the vertex of the cone being a point on that cubic. If the two assumed points be corresponding points on the Hessian, P_{12} vanishes identically; for the equation of the polar plane, with respect to a cone, of its vertex vanishes identically. Hence *the polar plane of any point of the line joining two corresponding points*

*on the Hessian passes through the intersection of the tangent planes to the Hessian at these points.** [More simply, the polar plane is the same as the polar plane with respect to its polar quadric, and therefore passes through $C'D'$ as at the end of Art. 526.] In any assumed plane we can draw three lines joining corresponding points on the Hessian; for the curve of the sixth degree considered in the last article meets the assumed plane in three pairs of corresponding points. The polar cubic then of the assumed plane will contain three right lines; as will otherwise appear from the theory of right lines on cubics, which we shall now explain.

[Ex. 1. The polar quadric of a double point on the Hessian with respect to the cubic is a pair of planes.

This is the converse of Art. 527*a*, and may be seen by expressing that the points (0, 0, 1, 0) and (0, 0, 0, 1) are corresponding points. By properly choosing the planes x and y the equation of cubic is

$$w^3 + w^2(lx + my) + hxyw + (xyz)^3 = 0.$$

The Hessian of this will have a double point at (0, 0, 0, 1) if a function, which is the discriminant of the polar quadric of this point, vanishes, disregarding an irrelevant factor which could only be zero if (0, 0, 1, 0) were a binode.

Ex. 2. The polar cubic of $\lambda x + \mu y + \nu z + \rho w + \sigma v = 0$ with respect to $\Sigma ax^3 = 0$ is $\Sigma(\lambda - \mu)^2\, cdezwv = 0$ which may be also written

$$\left\{\Sigma\frac{\lambda}{ax}\right\}^2 - \left(\Sigma\frac{1}{ax}\right)\left(\Sigma\frac{\lambda^2}{ax}\right) = 0,$$

and from this form the results of Art. 528 are obvious. In particular the four nodes are the correspondents of the points where the Hessian is met by $\Sigma\lambda^2 x = \Sigma\lambda x = 0$, and if $(x_r \ldots v_r)$ be these latter points the line of contact of the cone whose vertex is at the point x_1 is the intersection of the given plane with

$$\frac{x_r}{x_1}x + \frac{y_r}{y_1}y + \frac{z_r}{z_1}z + \frac{w_r}{w_1}w + \frac{v_r}{v_1}v = 0$$

r having the values 2, 3, or 4.

Ex. 3. The six points in which the sextic of Art. 528 meets the plane are the intersections of the four lines of contact of the polar quadrics of the nodes; and opposite intersections are corresponding points on the Hessian.

Let the double points be A, B, C, D, and let the corresponding points be $A'B'C'D'$, and the lines of contact of the cones be α, β, γ, δ. A', B', C', D' lie on the line $\Sigma\lambda x = 0$ $\Sigma\lambda^2 x = 0$, and $ABCD$ is the self-conjugate tetrahedron of

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* Steiner says that there are one hundred lines such that the polar plane of any point of one of them passes through a fixed line, but I believe that his theorem ought to be amended as above.

the system of polar quadrics of points on this line. The polar plane of B with respect to the polar quadric of A' is the plane ACD, which must be identical with the polar plane of A' with respect to the polar quadric of B. But this latter plane contains the line β. Thus β lies in the plane ACD and similarly α in the plane BCD, so that CD meets the intersection of α and β, say the point $\alpha\beta$. But CD lies altogether in the polar cubic, so that $\alpha\beta$ is in the polar cubic. By the argument just used, we can see that both the polar plane of A and that of B with respect to the polar quadric of $\alpha\beta$ must both be the plane $\Sigma\lambda x$, and therefore this quadric is a cone; and since it must pass through all the intersections of the polar quadrics of C and D and also have its vertex on $\Sigma\lambda x$ (because $\alpha\beta$ is on the polar cubic), its vertex must be $\gamma\delta$. So that $\alpha\beta$ and $\gamma\delta$ are corresponding points on the Hessian.]

529*a*. It is known that in a plane cubic the polar line, with respect to the Hessian, of any point on the curve, meets on the curve the tangent at that point. Clebsch has given as the corresponding theorem for surfaces, *The polar plane, with respect to the Hessian, of any point on the cubic, meets the tangent plane at that point, in the line which joins the three points of inflexion of the section by the tangent plane.* It will be remembered that the section by a tangent plane is a cubic having a double point, and therefore having only three points of inflexion lying on a line. If w be this line, xy the double point, the equation of such a curve may be written

$$x^3 + y^3 + 6xyw = 0.$$

Writing the equation of the surface (the tangent plane being z), $x^3 + y^3 + 6xyw + zu = 0$, where u is a complete function of the second degree $u = dz^2 + 6lxw + 6myw + 3nzw + \&c.$, of which we have only written the terms we shall actually require; and working out the equation of the Hessian, we find the terms below the second degree in x, y, z to be $d^2w^4 + d(n - 2lm)zw^3$. The polar plane then of the Hessian with respect to the point xyz is $4dw + (n - 2lm)z$, which passes through the intersection of zw, as was to be proved.

If the tangent plane $z = 0$ pass through one of the right lines on the cubic, the section by it consists of the right line x and a conic, and may be written $x^3 + 6xyw = 0$; and, as before, the polar plane of the point xyz with respect to the Hessian passes through the line zw, a theorem which may be geometrically

stated as follows: *When the section by the tangent plane is a line and a conic, the polar plane, with respect to the Hessian, of either point in which the line meets the conic, passes through the tangent to the conic at the other point.* If the tangent plane passes through two right lines on the cubic, the section reduces to xyw, and the polar plane still passes through zw, that is to say, through the third line in which the plane meets the cubic. If the point of contact is a cusp, it is proved in like manner that the line through which the polar plane passes is the line joining the cusp to the single point of inflexion of the section.

The conclusions of this article may be applied with a slight modification to surfaces of higher degree than the third: for if we add to the equation of the surface with which we have worked, terms of higher degree in xyz than the third, these will not affect the terms in the equation of the Hessian which are below the second degree in x, y, z. And the theorem is that the polar plane, with respect to the Hessian, of any point on a surface intersects the tangent plane at that point, in the line joining the points of inflexion of the section, by the tangent plane, of the polar cubic of the same point.

THE RIGHT LINES ON A CUBIC.

530. We said (Art. 49) that a cubical surface necessarily contains right lines, and we now enquire how many in general lie on the surface.* In the first place it is to be observed that if a right line lie on the surface, every plane through it is a double tangent plane because it meets the surface in a right line and conic; that is to say, in a section having two double points. The planes then joining any point to the right lines on the surface are double tangent planes to the surface, and

* The theory of right lines on a cubical surface was first studied in the year 1849, in a correspondence between Prof. Cayley and me, the results of which were published, *Cambridge and Dublin Mathematical Journal*, vol. IV. pp. 118, 252. Prof. Cayley first observed that a definite number of right lines must lie on the surface; the determination of that number as above, and the discussions in Art. 533 were supplied by me.

therefore also double tangent planes to the tangent cone whose vertex is that point. But we have seen (Art. 519) that the number of such double tangent planes is *twenty-seven.*

This result may be otherwise established as follows: let us suppose that a cubic contains one right line, and let us examine in how many ways a plane can be drawn through the right line, such that the conic in which it meets the surface may break up into two right lines. Let the right line be wz; let the equation of the surface be $wU = zV$; let us substitute $w = \mu z$, divide out by z, and then form the discriminant of the resulting quadric in x, y, z. Now in this quadric it is seen without difficulty that the coefficients of x^2, xy, and y^2 only contain μ in the first degree; that those of xz and yz contain μ in the second degree, and that of z^2 in the third degree. It follows hence that the equation obtained by equating the discriminant to nothing is of the fifth degree in μ; and therefore that *through any right line on a cubical surface can be drawn five planes, each of which meets the surface in another pair of right lines;* and, consequently, *every right line on a cubic is intersected by ten others.* Consider now the section of the surface by one of the planes just referred to. Every line on the surface must meet in some point the section by this plane, and therefore must intersect some one of the three lines in this plane. But each of these lines is intersected by eight in addition to the lines in the plane; there are therefore twenty-four lines on the cubic besides the three in the plane; that is to say, *twenty-seven in all.*

We shall hereafter show how to form the equation of a surface of the ninth degree meeting the given cubic in those lines.

531. Since the equation of a plane contains three independent constants, a plane may be made to fulfil any three conditions, and therefore a finite number of planes can be determined which shall touch a surface in three points. We can now determine this number in the case of a cubical surface. We have seen that through each of the twenty-seven

lines can be drawn five triple tangent planes: for every plane intersecting in three right lines touches at the vertices of the triangle formed by them, these being double points in the section. The number 5×27 is to be divided by three, since each of the planes contains three right lines; *there are therefore in all forty-five triple tangent planes.*

532. *Every plane through a right line on a cubic is obviously a double tangent plane; and the pairs of points of contact form a system in involution.* Let the axis of z lie on the surface, and let the part of the equation which is of the first degree in x and y be $(az^2+bz+c)x+(a'z^2+b'z+c')y$; then the two points of contact of the plane $y=\mu x$ are determined by the equation

$$(az^2+bz+c)+\mu(a'z^2+b'z+c)=0,$$

but this denotes a system in involution (*Conics*, Art. 342). It follows hence, from the known properties of involution, that two planes can be drawn through the line to touch the surface in two coincident points; that is to say, which cut it in a line and a conic touching that line. The points of contact are evidently the points where the right line meets the parabolic curve on the surface. It was proved (Art. 287) that the right line touches that curve. The two points then, where the line touches the parabolic curve, together with the points of contact of any plane through it, form a harmonic system. Of course the two points where the line touches the parabolic curve may be imaginary.

533. The number of right lines may also be determined thus. The form $ace=bdf$ (where a, b, &c., represent planes) is one which implicitly involves nineteen independent constants, and therefore is one into which the general equation of a cubic may be thrown.* This surface obviously contains nine lines (ab, cd, &c.). Any plane then $a=\mu b$ which meets the surface in right lines meets it in the same lines in which it

* It will be found in one hundred and twenty ways. [See Henderson's work cited in note to Art. 523.]

meets the hyperboloid $\mu ce = df$. The two lines are therefore generators of different species of that hyperboloid. One meets the lines cd, ef, and the other the lines cf, de. And, since μ has three values, [other than 0 and ∞,] there are three lines which meet ab, cd, ef. The same thing follows from the consideration that the hyperboloid determined by these lines must meet the surface in three more lines (Art. 345).

Now there are clearly six hyperboloids, ab, cd, ef; ab, cf, de, &c., which determine eighteen lines in addition to the nine with which we started, that is to say, as before, twenty-seven in all.

If we denote each of the eighteen lines by the three which it meets, the twenty-seven lines may be enumerated as follows: there are the original nine ab, ad, af, cb, cd, cf, eb, ed, ef; together with $(ab.cd.ef)_1$, $(ab.cd.ef)_2$, $(ab.cd.ef)_3$, and in like manner three lines of each of the forms $ab.cf.de$, $ad.bc.ef$, $ad.be.cf$, $af.bc.de$, $af.be.cd$. The five planes which can be drawn through any of the lines ab are the planes a and b, meeting the surface respectively in the pairs of lines ad, af; bc, be; and the three planes which meet the surface in $(ab.cd.ef)_1$, $(ab.cf.de)_1$; $(ab.cd.ef)_2$, $(ab.cf.de)_2$; $(ab.cd.ef)_3$, $(ab.cf.de)_3$. The five planes which can be drawn through any of the lines $(ab.cd.ef)_1$, meet the surface in the pairs of lines, ab, $(ab.cf.de)_1$; cd, $(af.cd.be)_1$; ef, $(ad.bc.ef)_1$; and in $(ad.be.cf)_2$, $(af.bc.de)_3$; $(ad.be.cf)_3$, $(af.bc.de)_2$.

534. Schläfli * has made a new arrangement of the lines which leads to a simpler notation, and gives a clearer conception how they lie. Writing down the two systems of six non-intersecting lines

$$ab,\ cd,\ ef,\ (ad.be.cf)_1,\ (ad.be.cf)_2,\ (ad.be.cf)_3,$$
$$cf,\ be,\ ad,\ (ab.cd.ef)_1,\ (ab.cd.ef)_2,\ (ab.cd.ef)_3,$$

it is easy to see † that each line of one system does not inter-

* *Quarterly Journal of Mathematics*, vol. II. p. 116.

† [$(ad.be.cf)_1$ and $(ab.cd.ef)_1$ are each met by $(ad.bc.ef)_1$, and the planes through the last and the former pair meet the cubic again in ad and ef, showing that the first two lines do not form one of the pairs lying in planes through the third.]

sect the line of the other system, which is written in the same vertical line, but that it intersects the five other lines of the second system. We may write then these two systems

$$a_1, a_2, a_3, a_4, a_5, a_6,$$
$$b_1, b_2, b_3, b_4, b_5, b_6,$$

which is what Schläfli calls a "double-six". It is easy to see from the previous notation that the line which lies in the plane of a_1, b_2, is the same as that which lies in the plane of a_2, b_1. Hence the fifteen other lines may be represented by the notation c_{12}, c_{34}, &c., where c_{12} lies in the plane of a_1, b_2, and there are evidently fifteen combinations in pairs of the six numbers 1, 2, &c. The five planes which can be drawn through c_{12} are the two which meet in the pairs of lines a_1b_2, a_2b_1, and those which meet in $c_{34}c_{56}$, $c_{35}c_{46}$, $c_{36}c_{45}$. There are evidently thirty planes which contain a line of each of the systems a, b, c; and fifteen planes which contain three c lines. It will be found that out of the twenty-seven lines can be constructed thirty-six "double-sixes". [They are the original, 15 of the type

$$a_1, b_1, c_{23}, c_{24}, c_{25}, c_{26},$$
$$a_2, b_2, c_{13}, c_{14}, c_{15}, c_{16},$$

and 20 of the type

$$a_1, a_2, a_3, c_{56}, c_{64}, c_{45},$$
$$c_{23}, c_{31}, c_{12}, b_4, b_5, b_6.]$$

535. We can now geometrically construct a system of twenty-seven lines which can belong to a cubical surface. We may start by taking arbitrarily any line a_1 and five others which intersect it, b_2, b_3, b_4, b_5, b_6. These determine a cubical surface, for if we describe such a surface through four of the points where a_1 is met by the other lines and through three more points on each of these lines, then the cubic determined by these nineteen points contains all the lines, since each line has four points common with the surface. Now if we are given four non-intersecting lines, we can in general draw two transversals which shall intersect

them all; for the hyperboloid determined by any three meets the fourth in two points through which the transversals pass (see Art. 53*d* and note to Art. 455). Through any four then of the lines b_3, b_4, b_5, b_6 we can draw in addition to the line a_1 another transversal a_2, which must also lie on the surface since it meets it in four points. In this manner we construct the five new lines a_2, a_3, a_4, a_5, a_6. If we then take another transversal meeting the four first of these lines, the theory already explained shows that it will be a line b_1 which will also meet the fifth. We have thus constructed a "double-six". We can then immediately construct the remaining lines by taking the plane of any pair a_1b_2, which will be met by the lines b_1, a_2 in points which lie on the line c_{12}.

536. Schläfli has made an analysis of the different species of cubics according to the reality of the twenty-seven lines. He finds thus five species: *A*. all the lines and planes real; *B*. fifteen lines and fifteen planes real; *C*. seven lines and five planes real; that is to say, there is one right line through which five real planes can be drawn, only three of which contain real triangles; *D*. three lines and thirteen planes real: namely, there is one real triangle through every side of which pass four other real planes; and, *E*, three lines and seven planes real.

I have also given (*Cambridge and Dublin Mathematical Journal*, vol. IV. p. 256) an enumeration of the modifications of the theory when the surface has one or more double points. It may be stated generally, that the cubic has always twenty-seven right lines and forty-five triple tangent planes, if we count a line or plane through a double point as two, through two double points as four, and a plane through three such points as eight. Thus if the surface has one double point, there are six lines passing through that point, and fifteen other lines, one in the plane of each pair. There are fifteen treble tangent planes not passing through the double point. Thus $2 \times 6 + 15 = 27$; $2 \times 15 + 15 = 45$.

Again, if the surface have four double points, the lines are

the six edges of the pyramid formed by the four points (6 × 4), together with three others lying in the same plane, each of which meets two opposite edges of the pyramid. The planes are the plane of these three lines 1, six planes each through one of these lines and through an edge (6 × 2), together with the four faces of the pyramid (4 × 8).

The reader will find the other cases discussed in the paper just referred to, and in a later memoir by Schläfli in the *Philosophical Transactions* for 1863.

[536*a*. Since the fourth edition of this book a considerable amount of literature has been published dealing with the lines on cubic surfaces. The principal results are presented in a connected form in the work by A. Henderson, mentioned in the note to Art. 523, a book which is also valuable for the bibliography of the subject which it contains. The theorem, generally known as Schläfli's theorem, that five lines a_2, a_3, a_4, a_5, a_6 constructed as in Art. 535 have a common transversal b_1, can of course be proved independently of the theory of cubic surfaces, and a variety of such proofs have been published. An elementary geometrical proof recently published by H. F. Baker * is contained in Exs. 1-3 below. R. Russell has communicated the proof contained in Ex. 4 which connects the theorem with Poncelet's theorem on coaxal circles; to him also is due the arrangement of the proof by line coordinates contained in Ex. 5.

Ex. 1. If ten lines be related as regards intersections like the lines a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_6, the tetrad of points a_1b_2, a_2b_1, a_3b_4, a_4b_3 are coplanar. This follows from the fact that if two quadrics have two intersecting lines common their remaining common points lie in a plane conic. Now the two hyperboloids determined by a_1, a_2, a_5 (or b_3, b_4, b_6) and a_3, a_4, a_5 (or b_1, b_2, b_6) have a_5 and b_6 common, and the points mentioned are common to both. Similarly a_1b_3, a_3b_1, a_2b_4, a_4b_2 are coplanar.

Ex. 2. If all the lines of Ex. 1 but b_1 are given, b_1 is uniquely determined. For b_1 is constructed by joining the point where a_2 meets the plane of a_1b_2, a_3b_4, a_4b_3, to the point where a_3 meets the plane of a_1b_3, a_2b_4, a_4b_2.

* *Proc. R. S.* (A), 84 (1911).

Ex. 3. Show from Ex. 2 that b_1 of Art. 535 meets a_2, a_3, a_4, a_5, a_6. The line b_1 as derived in Ex. 2 from the nine lines $a_1, a_2, a_3, a_4, b_2, b_3, b_4, a_5, a_6$ is clearly the same as that derived from the first seven and a_6, b_5.

Ex. 4 If a skew hexagon can be drawn whose sides in order are alternately generators of two quadrics U and V, any number of such hexagons can be drawn.

The six planes of pairs of consecutive sides of the hexagon are common tangent planes to U and V and therefore their traces in a principal plane are tangent to the conic in which the common tangent developable intersects that plane (Art. 216) and form a hexagon whose vertices are alternately on two conics and whose sides touch a third conic touched by the common tangents of the first two. Reciprocate Poncelet's theorem which states that if all the sides but one of a polygon inscribed in a circle touch coaxal circles, the envelope of the remaining side is another circle of the system; we then see that any number of such plane hexagons can be described and hence any number of the skew hexagons. Schläfli's theorem at once follows.

Ex. 5. Two quadrics U and V which each have with a third quadric W the relation

$$\Theta\Theta' - 4\,\Delta\Delta' = 0$$

are related as in the preceding example (see Schur, *Math. Ann.*, XVIII, or Bennett, London Math. Soc. (2) IX.).

Ex. 6. Prove Schlafli's theorem by the use of line coordinates.

Let (12345) denote the determinant of the fifth order whose constituents are $p_\alpha s_\beta + q_\alpha t_\beta + r_\alpha u_\beta + s_\alpha p_\beta + t_\alpha q_\beta + u_\alpha r_\beta$ and which is zero when the five lines have a common transversal. (Note to Art. 455.)

Let p_α, &c., be the coordinates of a_α, and p'_α, &c., those of b_α.

It is required to prove that (23456) = 0 is a consequence of the equation (2′3′4′5′6′) = 0 and the fact that the lines a_α, b_β ($\alpha \neq \beta$ and $\beta \neq 1$) intersect.

Now generally (12345) = 0 regarded as an equation connecting the coordinates of 5 expresses the fact that 5 meets one of the two common transversals of 1234 and therefore regarded as a quadratic function of p_5, &c., (12345) is resolvable into two linear factors. Applying this to (23452′) and remembering that $\overline{32'} = \overline{42'} = \overline{52'} = 0$ we get

$$\overline{34}\;\overline{45}\;\overline{53}\;\overline{22'}^2 = \lambda\,\overline{2'1'}\;\overline{2'6'} \text{ and similarly}$$
$$\overline{24}\;\overline{45}\;\overline{52}\;\overline{33'}^2 = \lambda\,\overline{3'1'}\;\overline{3'6'}$$
$$\overline{34}\;\overline{46}\;\overline{63}\;\overline{22'}^2 = \mu\,\overline{2'1'}\;\overline{2'5'}$$
$$\overline{24}\;\overline{46}\;\overline{62}\;\overline{33'}^2 = \mu\,\overline{3'1'}\;\overline{3'5'}$$

and so on. Substitute from these for the various constituents of (2′3′4′5′6′) = 0 and we find (23456) $k = 0$ where k is a non-vanishing factor.]

[537. There is an intimate connection between the theory of lines on a cubic surface and the bitangents of a plane quartic which may be exhibited by taking a point O on the cubic, drawing the tangent cone from O to the cubic, and considering the plane quartic in which this cone is intersected

by any plane, L, parallel to the tangent plane, P, to the cubic at O. Since the planes joining O to the lines on the surface are double tangent planes to the cone, they meet L in 27 of the bitangents of the quartic. The remaining bitangent is the line at infinity in L, its points of contact being given by the directions of the two asymptotic lines at O.

Any three coplanar lines on the cubic project into three bitangents which, with that at infinity, form a set of four bitangents whose points of contact lie on a conic. For if xw is a line on the surface and O is the point xyz the equation of the surface may be written

$$xU+2wV+w^2P=0;$$

the tangent cone is then $V^2-xUP=0$, which is touched twice by the six plane-pairs of the system $k^2xP+2kU+V=0$, and clearly these six pairs project into a group as explained in *Higher Plane Curves*, Art. 354.

But one of these pairs is xP and the others are easily seen to be the planes joining O to the lines on the cubic coplanar with xw. Denoting the bitangent at infinity by d and the others by the symbols of the corresponding lines on the cubic, the group of a_1 and d are the five pairs $b_r c_{1r}$; that of b_1 and d the five pairs $a_r c_{1r}$; whence $a_1 b_r$ and $b_1 a_r$ belong to the same group (that of dc_{1r}), and hence $a_r b_r$ belongs to the group of $a_1 b_1$, or in other words, *every double-six of lines on the cubic projects into twelve bitangents of a plane quartic belonging to the same group.*]

INVARIANTS AND COVARIANTS OF A CUBIC.

538. We shall in this section give an account of the principal invariants, covariants, &c., that a cubic can have. We only suppose the reader to have learned from the *Lessons on Higher Algebra*, or elsewhere, some of the most elementary properties of these functions. An invariant of the equation of a surface is a function of the coefficients, whose vanishing expresses some permanent property of the surface, as for example that it has a nodal point. A covariant, as for

example the Hessian, denotes a surface having to the original surface some relation which is independent of the choice of axes. A contravariant is a relation between α, β, γ, δ, expressing the condition that the plane $\alpha x + \beta y + \gamma z + \delta w$ shall have some permanent relation to the given surface, as for example that it shall touch the surface. The property of which we shall make the most use in this section is that proved (*Lessons on Higher Algebra*, Art. 139), viz. that if we substitute in a contravariant for α, β, &c., $\frac{d}{dx}$, $\frac{d}{dy}$, &c., and then operate on either the original function or one of its covariants, we shall get a new covariant, which will reduce to an invariant if the variables have disappeared from the result. In like manner, if we substitute in any covariant for x, y, &c., $\frac{d}{d\alpha}$, $\frac{d}{d\beta}$, &c., and operate on a contravariant, we get a new contravariant or invariant.

Now, in discussing these properties of a cubic we mean to use Sylvester's canonical form, in which it is expressed by the sum of five cubes. We have calculated for this form the Hessian (Art. 527), and there would be no difficulty in calculating other covariants for the same form. It remains to show how to calculate contravariants in the same case. Let us suppose that when a function U is expressed in terms of four independent variables, we have got any contravariant in α, β, γ, δ; and let us examine what this becomes when the function is expressed by five variables connected by a linear relation. But obviously we can reduce the function of five variables to one of four, by substituting for the fifth its value in terms of the others, viz. $w = -(x+y+z+v)$. To find then the condition that the plane $\alpha x + \beta y + \gamma z + \delta v + \epsilon w$ may have any assigned relation to the given surface, is the same problem as to find that the plane $(\alpha-\epsilon)x + (\beta-\epsilon)y + (\gamma-\epsilon)z + (\delta-\epsilon)v$ may have the same relation to the surface, its equation being expressed in terms of four variables; so that the contravariant in five letters is derived from that in four by substituting $\alpha-\epsilon$, $\beta-\epsilon$, $\gamma-\epsilon$, $\delta-\epsilon$ respectively for α, β, γ, δ. Every

contravariant in five letters is therefore a function of the differences between α, β, γ, δ, ϵ. This method will be better understood from the following example:—

Ex. The equation of a quadric is given in the form

$$ax^2 + by^2 + cz^2 + dv^2 + ew^2 = 0,$$

where $x + y + z + v + w = 0$; to find the condition that $\alpha x + \beta y + \gamma z + \delta v + \epsilon w$ may touch the surface. If we reduce the equation of the quadric to a function of four variables by substituting for w its value in terms of the others, the coefficients of x^2, y^2, z^2, v^2 are respectively $a + e$, $b + e$, $c + e$, $d + e$, while every other coefficient becomes e. If now we substitute these values in the equation of Art. 79, the condition that the plane $\alpha x + \beta y + \gamma z + \delta v$ may touch, becomes

$$\alpha^2(bcd + bce + cde + dbe) + \beta^2(cda + cde + dae + ace) + \gamma^2(dab + dae + abe + bde)$$
$$+ \delta^2(abc + abe + bce + cae) - 2e(ad\beta\gamma + bd\gamma\alpha + cd\alpha\beta + bc\alpha\delta + ca\beta\delta + ab\gamma\delta) = 0.$$

Lastly, if we write in the above for α, β, &c., $\alpha - \epsilon$, $\beta - \epsilon$, &c., it becomes

$$bcd(\alpha - \epsilon)^2 + cda(\beta - \epsilon)^2 + dab(\gamma - \epsilon)^2 + abc(\delta - \epsilon)^2 + bce(\alpha - \delta)^2 + cae(\beta - \delta)^2$$
$$+ abe(\gamma - \delta)^2 + ade(\beta - \gamma)^2 + bde(\alpha - \gamma)^2 + cde(\alpha - \beta)^2 = 0,$$

a contravariant which may be briefly written $\Sigma cde(\alpha - \beta)^2 = 0$.

539. We have referred to the theorem that when a contravariant in four letters is given, we may substitute for α, β, γ, δ differential symbols with respect to x, y, z, w; and that then by operating with the function so obtained on any covariant we get a new covariant. Suppose now that we operate on a function expressed in terms of five letters x, y, z, v, w. Since x appears in this function both explicitly and also where it is introduced in w, the differential with respect to x is $\frac{d}{dx} + \frac{dw}{dx}\frac{d}{dw}$, or, in virtue of the relation connecting w with the other variables, $\frac{d}{dx} - \frac{d}{dw}$. Hence, a contravariant in four letters is turned into an operating symbol in five by substituting for

$$\alpha, \beta, \gamma, \delta;\ \frac{d}{dx} - \frac{d}{dw},\ \frac{d}{dy} - \frac{d}{dw},\ \frac{d}{dz} - \frac{d}{dw},\ \frac{d}{dv} - \frac{d}{dw}.$$

But we have seen in the last article that the contravariant in five letters has been obtained from one in four, by writing for α, $\alpha - \epsilon$, &c. It follows then immediately that *if in any contravariant in five letters we substitute for* α, β, γ, δ, ϵ, $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$, $\frac{d}{dv}$, $\frac{d}{dw}$, *we obtain an operating symbol, with*

which operating on the original function, or on any covariant, we obtain a new covariant or invariant. The importance of this is that when we have once found a contravariant of the form in five letters we can obtain a new covariant without the laborious process of recurring to the form in four letters.

Ex. We have seen that $\Sigma cde(\alpha - \beta)^2$ is a contravariant of the form

$$ax^2 + by^2 + cz^2 + dv^2 + ew^2.$$

If then we operate on the quadric with $\Sigma cde\left(\frac{d}{dx} - \frac{d}{dy}\right)^2$, the result, which only differs by a numerical factor form

$$bcde + cdea + deab + eabc + abcd,$$

is an invariant of the quadric. It is in fact its discriminant, and could have been obtained from the expression, Art. 67, by writing, as in the last article, $a + e$, $b + e$, $c + e$, $d + e$ for a, b, c, d, and putting all the other coefficients equal to e.

540. In like manner it is proved that we may substitute in any covariant function for x, y, z, v, w, differential symbols with regard to $\alpha, \beta, \gamma, \delta, \epsilon$, and that operating with the function so obtained on any contravariant we get a new contravariant. In fact if we first reduce the function to one of four variables, and then make the differential substitution, which we have a right to do, we have substitnted for

$$x, y, z, v, w; \ \frac{d}{d\alpha}, \frac{d}{d\beta}, \frac{d}{d\gamma}, \frac{d}{d\delta}, \text{ and } -\left(\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma} + \frac{d}{d\delta}\right).$$

But since the contravariant in five letters was obtained from that in four by writing $\alpha - \epsilon$ for α, &c., it is evident that the differentials of both with regard to $\alpha, \beta, \gamma, \delta$ are the same, while the differential of that in five letters with respect to ϵ is the negative sum of the differentials of that in four letters with respect to $\alpha, \beta, \gamma, \delta$. But this establishes the theorem. By this theorem and that in the last article we can, being given any covariant and contravariant, generate another, which again, combined with the former, gives rise to new ones without limit.

541. The polar quadric of any point with regard to the cubic $ax^3 + by^3 + cz^3 + dv^3 + ew^3$ is

$$ax'x^2 + by'y^2 + cz'z^2 + dv'v^2 + ew'w^2 = 0,$$

Now the Hessian is the discriminant of the polar quadric. Its equation therefore, by Ex., Art. 539, is $\Sigma bcdeyzvw = 0$, as was already proved, Art. 527. Again, what we have called (Art. 528) the polar cubic of a plane

$$\alpha x + \beta y + \gamma z + \delta v + \epsilon w,$$

being the condition that this plane should touch the polar quadric is (by Ex., Art. 538) $\Sigma cdezvw\ (\alpha - \beta)^2 = 0$. This is what is called a mixed concomitant, since it contains both sets of variables x, y, &c., and α, β, &c.

If now we substitute in this for α, β, &c., $\frac{d}{dx}$, $\frac{d}{dy}$, &c., and operate on the original cubic, we get the Hessian; but if we operate on the Hessian we get a covariant of the fifth order in the variables, and the seventh in the coefficients, to which we shall afterwards refer as Φ,

$$\Phi = abcde\Sigma abx^2y^2z.$$

In order to apply the method indicated (Arts. 539, 540) it is necessary to have a contravariant; and for this purpose I have calculated the contravariant σ, which occurs in the equation of the reciprocal surface, which, as we have already seen, is of the form $64\sigma^3 = \tau^2$. The contravariant σ expresses the condition that any plane $\alpha x + \beta y +$ &c. should meet the surface in a cubic for which Aronhold's invariant S vanishes. It is of the fourth degree both in α, β, &c., and in the coefficients of the cubic. In the case of four variables the leading term is α^4 multiplied by the S of the ternary cubic got by making $x = 0$ in the equation of the surface. The remaining terms are calculated from this by means of the differential equation (*Lessons on Higher Algebra*, Art. 150). The form being found for four variables, that for five is calculated from it as in Art. 538. I suppress the details of the calculation, which, though tedious, present no difficulty. The result is

$$\sigma = \Sigma abcd(\alpha - \epsilon)\ (\beta - \epsilon)\ (\gamma - \epsilon)\ (\delta - \epsilon) \quad . \quad [1].$$

For facility of reference I mark the contravariants with numbers between brackets, and the covariants by numbers

between parentheses, the cubic itself and the Hessian being numbered (1) and (2). We can now, as already explained, from any given covariant and contravariant, generate a new one, by substituting in that in which the variables are of lowest dimensions, differential symbols for the variables, and then operating on the other. The result is of the difference of their degrees in the variables, and of the sum of their degrees in the coefficients. If both are of equal dimensions, it is indifferent with which we operate. The result in this case is an invariant of the sum of their degrees in the coefficients. The results of process are given in the next article.

542. (*a*) Combining (1) and [1], we expect to find a contravariant of the first degree in the variables, and the fifth in the coefficients; but this vanishes identically.

(*b*) (2) on [1] gives an invariant to which we shall refer as invariant A,

$$A = \Sigma b^2c^2d^2e^2 - 2abcde\Sigma abc.$$

If A be expressed by the symbolical method explained (*Lessons on Higher Algebra*, XIV., XIX.), its expression is

$$(1235)\,(1246)\,(1347)\,(2348)\,(5678)^2.$$

(*c*) Combining [1] with the square of (1) we get a covariant quadric of the sixth order in the coefficients

$$abcde(ax^2 + by^2 + cz^2 + dv^2 + ew^2) \quad . \quad (3),$$

which expressed symbolically is (1234) (1235) (1456) (2456).

(*d*) (3) on [1] gives a contravariant quadric

$$a^2b^2c^2d^2e^2\Sigma(\alpha - \beta)^2 \quad . \quad . \quad . \quad [2].$$

(*e*) [2] on (1) gives a covariant plane of the eleventh order in the coefficients

$$a^2b^2c^2d^2e^2(ax + by + cz + dv + ew) \quad . \quad (4).$$

(*f*) (3) on [2] gives an invariant B,

$$a^3b^3c^3d^3e^3(a + b + c + d + e).$$

(*g*) Combining with (3) the mixed concomitant (Art. 541) we get a covariant cubic of the ninth order in the coefficients

$$abcde\Sigma cde\,(a+b)\,zvw \quad . \quad . \quad (5).$$

(h) Combining (5) and [1] we have a linear contravariant of the thirteenth order in the coefficients

$$abcde\Sigma(a-b)(\alpha-\beta)\{(a+b)c^2d^2e^2 - abcde(cd+de+ec)\}\ [3].$$

It seems unnecessary to give further details as to the steps by which particular concomitants are found, and we may therefore sum up the principal results.

543. It is easy to see that every invariant is a symmetric function of the quantities a, b, c, d, e. If then we denote the sum of these quantities, of their products in pairs, &c., by p, q, r, s, t, every invariant can be expressed in terms of these five quantities, and therefore in terms of the five following fundamental invariants, which are all obtained by continuing the process exemplified in the last article,

$$A = s^2 - 4rt,\ B = t^3p,\ C = t^4s,\ D = t^6q,\ E = t^8;$$

whence also $$C^2 - AE = 4t^9r.$$

We can, however, form skew invariants which cannot be rationally expressed in terms of the five fundamental invariants, although their squares can be rationally expressed in terms of these quantities. The simplest invariant of this kind is got by expressing in terms of its coefficients the discriminant of the equation whose roots are a, b, c, d, e. This, it will be found, gives in terms of the fundamental invariants A, B, C, D, E an expression for t^{36} multiplied by the product of the squares of the differences of all the quantities a, b, &c. This invariant being a perfect square, its square root is an invariant F of the one-hundredth degree. Its expression in terms of the fundamental invariants is given, *Philosophical Transactions*, 1860, p. 233.

The discriminant of the cubic can easily be expressed in terms of the fundamental invariants. It is obtained by eliminating the variables between the four differentials with respect to x, y, z, v, that is to say,

$$ax^2 = by^2 = cz^2 = dv^2 = ew^2.$$

Hence x^2, y^2, &c., are proportional to $bcde$, $cdea$, &c. Substituting then in the equation $x+y+z+v+w=0$, we get the discriminant

$$\sqrt{(bcde)} + \sqrt{(cdea)} + \sqrt{(deab)} + \sqrt{(eabc)} + \sqrt{(abcd)} = 0.$$

Clearing of radicals, the result, expressed in terms of the principal invariants, is

$$(A^2 - 64B)^2 = 16384(D + 2AC).$$

544. The cubic has four fundamental covariant planes of the orders 11, 19, 27, 43 in the coefficients, viz.

$$L = t^2\Sigma ax,\ L' = t^3\Sigma bcdex,\ L'' = t^5\Sigma a^2x,\ L''' = t^8\Sigma a^3x.$$

Every other covariant, including the cubic itself, can, in general, be expressed in terms of these four, the coefficients being invariants. The condition that these four planes should meet in a point, is the invariant F of the one-hundredth degree.

There are linear contravariants, the simplest of which, of the thirteenth degree, has been already given; the next being of the twenty-first, $t^4\Sigma(a-b)(\alpha-\beta)$; the next of the twenty-ninth, $t^5\Sigma cde(a-b)(\alpha-\beta)$, &c.

There are covariant quadrics of the sixth, fourteenth, twenty-second, &c., orders; and contravariants of the tenth, eighteenth, &c., the order increasing by eight.

There are covariant cubics of the ninth order $t\Sigma cde(a+b)zvw$, and of the seventeenth, $t^3\Sigma a^2x^3$, &c.

If we call the original cubic U, and this last covariant V, since if we form a covariant or invariant of $U + \lambda V$, the coefficients of the several powers of λ are evidently covariants or invariants of the cubic: it follows that, given any covariant or invariant of the cubic we are discussing, we can form from it a new one of the degree sixteen higher in the coefficients, by performing on it the operation

$$t^3\left(a^2\frac{d}{da} + b^2\frac{d}{db} + c^2\frac{d}{dc} + d^2\frac{d}{dd} + e^2\frac{d}{de}\right).$$

Of higher covariants we only think it necessary here to mention one of the fifth order, and fifteenth in the coefficients t^3xyzvw, which gives the five fundamental planes; and one of the ninth order, Θ the locus of points whose polar planes with respect to the Hessian touch their polar quadrics with respect to U. Its equation is expressed by the determinant,

Art. 79, using α, β, &c., to denote the first differential coefficients of the Hessian with respect to the variables, and a, b, &c., the second differential coefficients of the cubic.

The equation of a covariant, whose intersection with the given cubic determines the twenty-seven lines, is $\Theta = 4\text{H}\Phi$, where Φ has the meaning explained, Art. 541. I verified this form, which was suggested to me by geometrical considerations, by examining the following form, to which the equation of the cubic can be reduced, by taking for the planes x and y the tangent planes at the two points where any of the lines meet the parabolic curve, and two determinate planes through these points for the planes, w, z,

$$z^2y + w^2x + 2xyz + 2xyw + ax^2y + by^2x + cx^2z + dy^2w = 0.$$

The part of the Hessian then which does not contain either x or y is z^2w^2; the corresponding part of Φ is $-2(cz^5 + dw^5)$, and of Θ is $-8w^2z^2(cz^5 + dw^5)$. The surface $\Theta - 4\text{H}\Phi$ has therefore no part which does not contain either x or y, and the line xy lies altogether on the surface, as in like manner do the rest of the twenty-seven lines.* Clebsch obtained the same formula directly, by the symbolical method of calculation, for which we refer to the *Lessons on Higher Algebra.*

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* This section is abridged from a paper which I contributed to the *Philosophical Transactions*, 1860, p. 229. Shortly after the reading of my memoir, and before its publication, there appeared two papers in Crelle's *Journal*, vol. LVIII., by Clebsch, in which some of my results were anticipated; in particular the expression of all the invariants of a cubic in terms of five fundamental invariants, and the expression given above for the surface passing through the twenty-seven lines. The method, however, which I pursued was different from that of Clebsch, and the discussion of the covariants, as well as the notice of the invariant F, I believe were new. Clebsch has expressed his last four invariants as functions of the coefficients of the Hessian. Thus the second is the invariant $(1234)^4$ of the Hessian, &c.

CHAPTER XVI.

SURFACES OF THE FOURTH DEGREE.

545. THE theory of the general quartic surface has hitherto been little studied. [Rohn has investigated the shapes of such surfaces as regards the number of closed ovals they may possess; Sisam the possibility of expressing their equations as the sum of the squares of five quadrics, and Schmidt the properties of their polar quadrics and the surfaces corresponding to the polar cubic of a plane with respect to a cubic (Art. 528).*] The quartic developable, or torse, has been considered, Art. 367. Other forms of quartics, to which much attention has been paid, are the ruled surfaces or scrolls which have been discussed by Chasles, Cayley, Schwarz, Cremona, [Segen and Williams]; † and quartics with a nodal conic which have been studied, in their general form, by Kummer, Clebsch, Korndörfer, [Segre, Zeuthen,] and others; and in the case where the nodal curve is the circle at infinity (under the names of cyclides and anallagmatic surfaces) by Casey, Darboux, Moutard, [Loria, Bertrand], and others.‡ In fact, in the classification of sur-

* Rohn, *Math. Ann.*, XXIX.; *Leipziger Berichte*, LXIII. 1911; Sisam, *Amer. Math. Soc. Bull.* (2), 14; Schmidt, Prize Dissertation, Breslau, 1908, "*Uber Zweite Polarflächen einer allgemein Fläche 4 Ordnung*".

† Chasles, *Comptes Rendus*, 1861; Cayley, *Phil. Trans.*, 1864, or *Collected Papers*, V. 201, and *Phil. Trans.*, 1869, or *Collected Papers*, VI. 312; Cremona, *Mem. di Bologna*, VIII. 1868; Segen, *Crelle*, CXII.; Williams, *Proc. Amer. Acad.*, 36.

‡ Kummer, *Berlin Monatsberichte*, July, 1863; *Crelle*, LXIV.; Clebsch, *Crelle*, LXIX.; Korndörfer, *Math. Ann.*, I., II., III.; Zeuthen, *Ann. di Mat.*, XIV.; Segre, *Math. Ann.*, XXIV.; Casey and Darboux as cited in note to Art. 515. See also list of memoirs given in Darboux's work. Loria, *Mem. Acc. Turin*, XXXVI. 1884; Bertrand, *Nouv. Ann. de M.* (3), 9 (Dupin's Cyclide).

faces according to their degree, the extent of the subject increases so rapidly with the degree, that the theory for example of the particular kind of quartics last mentioned may be regarded as co-extensive with the entire theory of cubics.

[Quartics with a nodal right line have been studied by Zimmerman, those with a triple point by Rohn, and those with isolated conical points by Cayley and Rohn, while the plane representation of the last class of surfaces and the treatment of curves lying on them by aid of hyperelliptic functions have been the subjects of papers by Remy and others.*

The six-nodal quartic (Weddle's surface) has been recently discussed by Baker, Bateman, and by Morley and Conner, and the Steinerian surfaces of several special quartics by Van der Vries.†

Steiner's quartic (Art. 523) is treated by Kummer and many others.‡

The sixteen-nodal quartic, known as Kummer's, is the subject of a recent book by Hudson, § who has summarised into a remarkably concise and readable volume much of the immense literature that has grown up in connection with this surface. The reader is referred to this work for references to the original memoirs and for fuller information about the surface than is contained in the sections at the end of this chapter.]

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* Zimmerman, Prize Dissertation, Breslau, 1904; Rohn, *Math. Ann.*, XXIV. (on triple-point quartics), and *Math. Ann.*, XXIX. (on nodal quartics); Cayley, *Three Memoirs on Quartics*, in vol. VII. of collected works; Remy, *Comptes Rendus*, 143, 148, 149, and on same subject, Traynard, *C.R.*, 140; Garnier, *C.R.*, 149; Chillemi, *Rend. Circ. Mat. Palermo*, 29; Maroni, *Lomb. Inst.* (2), 38, and Fano, *ibid.*, 39.

† Baker, *Proc. Lond. Math. Soc.* (2), 1; Bateman, *ibid.* (2), 3; Morley and Conner, *Amer. J. of M.*, 31; Van der Vries, *ibid.*, 32.

‡ Kummer and Schröter, *Berl. Monatsb.*, 1863; *Crelle*, LXIV.; Cremona, *Crelle*, LXIII.; Cayley, *ibid.*, LXIV.; Clebsch, *ibid.*, LXVII.; Gerbaldi, *La Superficie di Steiner*, Turin, 1881; Vahlen, *Act. Math.*, XIX.; Lacour, *Nou. Ann.* (3), XVII.; Cotty, *ibid.* (4), VIII.; Montesano, *Nap. Rend.* (3), V.

§ *Kummer's Quartic Surface*, by R. W. T. Hudson. Cambridge, 1905.

QUARTICS WITH SINGULAR LINES—SCROLLS.

546. The highest singularity which a quartic can possess is a triple line, which is necessarily a right line. Every such surface is a scroll, for it evidently contains an infinity of right lines, since every plane section through the triple line consists of that line counted thrice and another line. The equation may be written in the form $u_4 = zu_3 + wv_3$, where u_4, u_3, v_3 are functions of the fourth and third orders respectively in x and y, and xy denotes the triple line. The three tangent planes at any point on the triple line are given by the equation $z'u_3 + w'v_3 = 0$. Forming the discriminant of this equation, we see that there are in general four points [known as "pinch" points] on the triple line, at which two of its tangent planes coincide. We may take z and w as planes passing each through one of these points, and x and y as the corresponding double tangent planes, when the equation becomes

$$u_4 = z(ax^3 + bx^2y) + w(cxy^2 + dy^3).$$

Further, by substituting for z, $z + \alpha x + \beta y$, and for w, $w + \gamma x + \delta y$, we can evidently determine α, β, γ, δ, so as to destroy the terms x^4, x^3y, y^3x, y^4 in u_4; and so, finally, reduce the equation to the form

$$mx^2y^2 = z(ax^3 + bx^2y) + w(cxy^2 + dy^3).$$

The planes z, w evidently touch the surface along the whole lengths of the lines zy, wx, respectively; and we see that the surface has four *torsal* generators, see note, Art. 522.

The surface may be generated according to the method of Art. 467, the directing curves being the triple line, and any two plane sections of the surface; that is to say, the directing curves are two plane quartics, each with a triple point, and the line joining the triple points, the quartics having common the four points in which each is met by the intersection of their planes. [If we put 1, 4, 4, 4, 3, 3 for m_1, m_2, m_3, a, β, γ in the formulæ of Art. 468, we get $m_2m_3 - a = 12$. This number includes the line considered nine times among the lines joining any point on itself to the apparent intersections of the quartics,

so that the relevant multiplicity is three as it should be.] But the generation is more simple if we take each plane section as one made by the plane of two generators which meet in the triple line. This will be a conic in addition to these lines; and the scroll is generated by a line whose directing curves are two non-intersecting conics, and a right line meeting both conics.

The equation of a quartic with a triple line may also be obtained by eliminating, between the equations of two planes, a parameter entering into one in the first, into the other in the third degree; for instance,

$$\lambda x + y = 0,\ \lambda^3 u + \lambda^2 v + \lambda w + z = 0;$$

that is to say, the generating line is the intersection of one of a series of planes through a fixed line with the corresponding one of a series of osculating planes to a twisted cubic, or tangent planes to a quartic torse. The four points where the torse meets the fixed line are the four pinch points already considered.

547. Returning to the equation

$$mx^2y^2 = z(ax^3 + bx^2y) + w(cxy^2 + dy^3)$$

there is an important distinction according as m does or does not vanish; or, in the form first given, according as u_4 is or is not capable of being expressed in the form $(\alpha x + \beta y)u_3 + (\gamma x + \delta y)v_3$. When m vanishes (II) the surface contains a right line zw which does not meet the triple line; otherwise (I) there is no such line. The existence of such a line implies a triple line on the reciprocal surface and *vice versa*. In fact, we have seen that every plane through the triple line contains one generator; to it will correspond in the reciprocal surface a line through every point of which passes one generator; that is to say, which is a simple line on the surface. Conversely, if a quartic scroll contain a director right line, every plane through it meets the surface in a right line and a cubic, and touches the surface in the three points where these intersect. Every plane through the right line therefore being a triple tangent plane, there will

correspond on the reciprocal surface a line every point of which is a triple point. In the case, therefore, where m vanishes the equation of the reciprocal is reducible to the same general form as that of the original.

In the general case (I) we can infer as follows the nature of the nodal curve in the reciprocal. At each point on the triple line can be drawn three generators. Consider the section made by the plane of any two; this will consist of two right lines and a conic through their intersection; and the plane will touch the surface at the two points where the lines are met again by the conic. Hence, at each point of the triple line three bitangent planes can be drawn to the scroll; and reciprocally every plane through the corresponding line meets the nodal curve of the reciprocal surface in three points. We infer then that this curve is a skew cubic, and we shall confirm this result by actually forming the equation of the reciprocal surface. It will be observed how the argument we have used is modified when the scroll has a simple director line, the three generators at any point of the triple line then lying all in one plane. If we substitute $y=\lambda x$ in the equation of the scroll, we see that any generator is given by the equations

$$y=\lambda x,\ m\lambda^2 x=(a+b\lambda)z+(c\lambda^2+d\lambda^3)w,$$

and joins the points

$$x=a+b\lambda,\ y=\lambda(a+b\lambda),\ z=m\lambda^2,\ w=0,$$
$$x=c+d\lambda,\ y=\lambda(c+d\lambda),\ z=0,\quad w=m.$$

The reciprocal line is therefore the intersection of

$$(x+\lambda y)\,(a+b\lambda)+m\lambda^2 z=0,\ (x+\lambda y)\,(c+d\lambda)+mw=0,$$

and the equation of the reciprocal is got by eliminating λ between these equations. But if we consider the scroll generated by the intersection of corresponding tangent planes to two cones

$$\lambda^2 x+\lambda y+z=0,\ \lambda^2 u+\lambda v+w=0,$$

this will be a quartic $(xw-uz)^2=(yw-zv)\,(xv-yu)$ which has a twisted cubic for a nodal line, since the three quadrics represented by the members of this equation have common a twisted cubic, as is evident by writing their equations in the

form $\frac{u}{x}=\frac{v}{y}=\frac{w}{z}$. In the case actually under consideration, the equation of the reciprocal is

$$\{m^2zw + mczx + mbyw + (bc - ad)xy\}^2$$
$$= \{mdzx + mczy + (bc - ad)y^2\} \{mbxw + amyw + (bc - ad)x^2\}.$$

This equation would become illusory if m vanished; and we must in that case (II) revert to the original form of the equations of a generator, which gives

$$y = \lambda x,\ (a + b\lambda)z + \lambda^2(c + d\lambda)w = 0.$$

The generator of the reciprocal scroll will be

$$\lambda y + x = 0,\ \lambda^2(c + d\lambda)z = (a + b\lambda)w,$$

and the reciprocal is obviously of like nature with the original. [Quartics of class II may be generated by a line meeting a skew cubic, one of its chords, and another line, the points where the chord meets the cubic being two of the pinch points.]

The two classes of scrolls we have examined each include two subforms according as either b or c, or both, vanish. In these cases the triple line has either one or two points at which all three tangent planes coincide. According to the mode of generation, noticed at the end of last article, the fixed line touches the torse, and either one pair or two pairs of the pinch points coincide.

548. Besides the two classes of quartic scrolls with a triple line, already mentioned, we count the following:—

III. u_3 and v_3 may have a common factor, which answers to the case $ad = bc$ in the equation already given: which is then reducible to the form

$$mx^2y^2 = (ax + by)\ (zx^2 + wy^2).$$

In this case also, in the method of Art. 546, the fixed line touches the torse. The generator of the scroll in one position coincides with the triple line, $ax + by$ being the corresponding tangent plane which osculates along its whole length. Also the equation of the reciprocal scroll being

$$(mzw + axz + byw)^2 = zw(ay + bx)^2,$$

we see that it has as nodal lines the plane conic $ay+bx$, $mzw+axz+byw$, and the right line zw which intersects that conic. This class contains as subform the case where $u_3+\lambda v_3$ includes a perfect cube. The equation may then be reduced to the form

$$my^4 = x(zx^2+wy^2),$$

the reciprocal of which is

$$(xz-mw^2)^2 = y^2zw.$$

IV. Again, u_3 and v_3 may have a pair of common factors and the equation is reducible to the form

$$x^2y^2 = (ax^2+bxy+cy^2)\ (xz+yw),$$

an equation which is easily seen by the same method, as before, to have a reciprocal of like form with itself.

V. Lastly, u_3 and v_3 may have common a square factor, the equation then taking the form

$$x^2y^2 = (ax+by)^2\ (xz+yw),$$

which is also its own reciprocal.* In this case two of the three sheets, which meet in the triple line, unite into a single cuspidal sheet. The case where u_3 and v_3 have three common factors need not be considered, as the surface would then be a cone. [Surfaces of classes III, IV, and V have respectively three, two, and one distinct pinch points.]

549. We come now to quartic scrolls with only double lines. If a quartic have a non-plane nodal line, it will ordinarily be a scroll. For take any fixed point on the nodal line, and there is only one condition to be fulfilled in order that the line joining this to any variable point on the nodal line may lie altogether in the surface, a condition which we can ordinarily fulfil by means of the disposable parameter which regulates the position of the variable point. There being thus an infinite series of right lines, the surface is a scroll. But a

* The first four classes enumerated answer to Cayley's ninth, third, twelfth, sixth, respectively; the last might be regarded as a subform of that preceding, but I have preferred to count it as a distinct class.

case of exception occurs, when the surface has three nodal right lines meeting in a point. Here the section by the plane of any two consists of these lines, each counted twice, and there is no intersecting line lying in the surface. This is Steiner's quartic mentioned note Art. 523 [see Art. 554*a*].

We consider now the other cases of quartics with nodal lines, commencing with those in which the line is of the third order. The case where the nodal lines are three right lines, no two of which are in the same plane, need not be considered, since it is easy to see that then the quartic is nothing else than the quadric, counted twice, generated by a line meeting these three director lines.

Let us commence with the case where the nodal line is a twisted cubic (VI and VII). Such a cubic may be represented by the three equations $xz - y^2 = 0$, $xw - yz = 0$, $yw - z^2 = 0$; the planes x and w being any two osculating planes of the cubic. The coordinates of any point on it may be taken as $x : y : z : w = \lambda^3 : \lambda^2 : \lambda : 1$. If the three quantities $xz - y^2$, $xw - yz$, $yw - z^2$ are called α, β, γ respectively, any quartic which has the cubic for a nodal line will be represented by a quadratic function of α, β, γ, say

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

Now consider the line joining two points on the cubic λ, μ; the coordinates of any point on it will be of the form $\lambda^3 + \theta\mu^3$, $\lambda^2 + \theta\mu^2$, $\lambda + \theta\mu$, $1 + \theta$. If we substitute these values in α, β, γ, they become, after dividing by the common factor $\theta(\lambda - \mu)^2$, $\lambda\mu$, $\lambda + \mu$, 1. Consequently the condition that the line should lie on the surface is

$$a\lambda^2\mu^2 + b(\lambda + \mu)^2 + c + 2f(\lambda + \mu) + 2g\lambda\mu + 2h\lambda\mu(\lambda + \mu) = 0.$$

Thus if either point be given, we have a quadratic to determine the position of the other; and we see that the surface is a scroll, and that through each point of the nodal line can be drawn two generators, each meeting the cubic twice. The six coordinates (Art. 57*a*) of the line joining the points λ, μ are easily seen to be (omitting a common factor $\lambda - \mu$)

$$\lambda^2 + \lambda\mu + \mu^2,\ \lambda + \mu,\ 1,\ \lambda\mu,\ -\lambda\mu(\lambda + \mu),\ \lambda^2\mu^2,$$

and as the condition just found is linear in these coordinates, we may say that a quartic scroll is generated by a line meeting a twisted cubic twice and whose six coordinates are connected by a linear relation, or, in other words, by the lines of a linear complex which join two points on a twisted cubic.

In fact, if p, q, r, s, t, u be the six coordinates, we have the relation

$$bp + 2fq + cr + (b + 2g)s - 2ht + au = 0.$$

We saw (Art. 57*c*) that a particular case of the linear relation between the six coordinates of a line is the condition that it shall intersect a fixed line; and from what was there said, and from what has now been stated, it follows immediately that all the generators of the scroll will meet a fixed line, provided the quantities multiplying p, q, &c., in the preceding equation be themselves capable of being the six coordinates of a line; that is to say (VII), provided the condition be fulfilled,

$$b(b + 2g) - 4fh + ac = 0.$$

When this condition is fulfilled, it appears, from Art. 547, that the reciprocal of the scroll will have a triple line, the reciprocal in fact belonging to the first class of scrolls with a triple line there considered.

550. In order to find the equation of the reciprocal in the general case VI, we observe that to the generator joining the points, whose coordinates are $\lambda^3, \lambda^2, \lambda, 1$; $\mu^3, \mu^2, \mu, 1$, will correspond on the reciprocal scroll the generator whose equations are

$$x\lambda^3 + y\lambda^2 + z\lambda + w = 0, \; x\mu^3 + y\mu^2 + z\mu + w = 0,$$

and the equation of the reciprocal is got by eliminating λ, μ between these equations and the relation already given connecting λ, μ. This elimination has been performed by Cayley; the work is too long to be here given, but the result is that the equation of the reciprocal scroll is of the same form and with the same coefficients as the original; so that the scroll which has been defined as generated by a line in involution twice meeting a skew cubic may also be defined as generated by a line in involution lying in two osculating planes

of a skew cubic. Thus then the fundamental division of scrolls with a nodal cubic is into scrolls whose reciprocals are of like form (VI), and scrolls whose reciprocals have a triple line (VII). It is to be noted that the general form of the equation of the reciprocal contains as a factor the quantity $b^2 + 2bg - 4fh + ac$, the vanishing of which implies that the scroll belongs to the latter class. The two classes of scrolls may be generated by a line twice meeting a skew cubic, and also meeting, in the one case, a conic twice meeting the cubic; in the other, a right line.*

551. If we put $\lambda = \mu$ in the equation of Art. 549, we obtain the points at which a generator will coincide with a tangent to the cubic; and this equation being of the fourth degree we see that the intersection of the scroll with the torse $4a\gamma - \beta^2 = 0$, of which the cubic is the cuspidal edge, is made up of the cubic [counting four times] together with four common generators. There will be four pinch-points on the cubic, these points being obtained by arranging the condition already obtained

$$\mu^2(a\lambda^2 + 2h\lambda + b) + 2\mu\{h\lambda^2 + (b+g)\lambda + f\} + b\lambda^2 + 2f\lambda + c = 0,$$

and forming the discriminant

$$(a\lambda^2 + 2h\lambda + b)(b\lambda^2 + 2f\lambda + c) = \{h\lambda^2 + (b+g)\lambda + f\}^2.$$

[The plane of a generator and the tangent line to the cubic is a tangent plane to the scroll, and this equation therefore expresses that the two tangent planes at λ coincide.] We might have so chosen our planes of reference that one of these four points should correspond to $\lambda = 0$, the other extremity of the generator through that point being $\mu = \infty$, and in this case $f = 0$, $b = 0$; or the equation of the scroll may always be transformed to the form

$$a\alpha^2 + c\gamma^2 + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

Or, again, by choosing the planes of reference so that two of the four points may be $\lambda = 0$, $\lambda = \infty$, the equation may be changed to the form $(a\alpha + b\beta + c\gamma)^2 = 4m^2\gamma\alpha$.

* These classes, my sixth and seventh, answer to Cayley's tenth and eighth.

We have a subform of the scroll, if either a or $c=0$ in this equation; for in this case two of the four pinch-points on the nodal curve coincide, the generator at this point being also a generator of the torse, and there is a common tangent plane to scroll and torse along this line.

A third of the pinch-points would unite if we had $b=m$; and if along with this condition we have both a and $c=0$, the surface is the torse $\beta^2-4\gamma\alpha=0$.

552. The next species of scrolls to be considered is when the nodal curve consists of a conic and right line (VIII and IX). The line necessarily meets the conic, which includes every point of the section of the scroll by its plane. This scroll may be generated by a line meeting two conics which have common the points in which each is met by the intersection of their planes, and also a line meeting one of the conics. [A plane through the nodal line will meet the quartic in a pair of right lines in addition, and for two positions of the plane these lines will coincide, or the plane will be a trope. Let $y=0$ be one of these planes and let $y=0$, $z=0$ be the coincident lines. Take $w=0$ as the plane of the conic and let the planes of x and z cut this plane in the tangents to the conic where y meets it.] Then it is easy to see that the most general equation of the scroll can be reduced to the form

$$(xz-y^2)^2+myw(xz-y^2)+w^2(axy+by^2)=0,$$

where $xz-y^2$, w is the nodal conic, xy the double line, and yz is one position of the generator. Take then any point on the conic, whose coordinates are λ^2, λ, 1, 0; and any point $z=\mu w$ on the line xy, and the line joining these points will lie altogether on the surface if

$$\lambda^2\mu^2+m\lambda\mu+a\lambda+b=0.$$

Thus two generators pass through any point of either nodal line or nodal conic. The reciprocal is got by eliminating between $\lambda^2x+\lambda y+z=0$, $\mu z+w=0$, and the preceding equation, and is

$$(bxz-w^2)^2-y(bxz-w^2)(by+mw-az)+xz(by+mw-az)^2=0,$$

which for b not equal 0 is a scroll of the same kind having the nodal conic, $bxz - w^2$, $by + mw - az$, and the nodal line zw; this is VIII. [There are two pinch-points on the line, namely, where it meets $(m \pm 2b^{\frac{1}{2}})z + aw$ and two on the conic, the points for which $\lambda = \infty$ or $a\lambda = m^2 - 4b$.] There is a subform when $m^2 = 4b$, that is to say, when the equation is of the form

$$(xz - y^2 + myw)^2 = aw^2xy.$$

[In this case there is but one pinch-point on the line and but one on the conic distinct from the point xyw.]

If $b = 0$ we have case IX; the reciprocal quartic has here a triple line and is of the third class already considered.* There is one pinch-point on the line and two on the conic.

553. The next case (X) is where the conic degenerates into a pair of lines, in other words, where there are two non-intersecting double lines, and a third cutting the other two. This class is a particular case of that next to be considered, viz. where the scroll is generated by a line meeting two non-intersecting right lines. If in any case two positions of the generator can coincide we have a double generator, and the scroll is that now under consideration. Thus, for example, the scroll generated by a line meeting two lines not in the same plane and also a conic is (Art. 467) of the fourth degree and has the two right lines as double lines; but two positions of the generator coincide with the line joining the points where the directing lines meet the plane of the conic, which is accordingly a third double line on the scroll. The general equation may be written as in the last article,

$$x^2z^2 + mxzyw + w^2(axy + by^2) = 0;$$

the line $x = \lambda y$, $z = \mu w$ will be a generator if

$$\lambda^2\mu^2 + m\lambda\mu + a\lambda + b = 0,$$

and the reciprocal is

$$y^2w^2 + mxzyw + xz^2\ (bx - ay) = 0,$$

* These two species, my eighth and ninth, are Cayley's seventh and eleventh respectively.

that is to say, is of the same nature as the original. This is Cayley's second species. As before, the form

$$(xz - yw)^2 = axyw^2$$

may be regarded as special.

[In the general case there are two pinch-points on each of the lines xy and zw, and in the special case there is one pinch-point on each of these lines, while every point of xw is a pinch-point, or in other words this line has become a cuspidal line. These species of scrolls are the subjects of the memoir by Segen quoted in the footnote to Art. 545.]

554. Next let us take the general case (XI) (Cayley's first species) where there are two non-intersecting double lines. This scroll may be generated by a line meeting a plane binodal quartic, and two lines, one through each node. When the quartic has a third node we have the species of last article. The most general equation is

$$x^2(az^2 + 2hzw + bw^2) + 2xy(a'z^2 + 2h'zw + b'w^2) + y^2(a''z^2 + 2h''zw + b''w^2) = 0,$$

the reciprocal of which is easily shown to be of like form. There are obviously four pinch-points on each line, and sub-forms may be enumerated according to the coincidence of two or more of these points.

But again, (XII) in the generation by the binodal quartic just mentioned two of the nodes may coalesce in a tacnode; and we have then a scroll with two coincident double lines (Cayley's fourth species), the general equation of which may be written

$$u_4 + (yw - xz)\, u_2 + (yw - xz)^2 = 0,$$

where u_4, u_2 are a binary quartic and quadratic in x and y; and the reciprocal is of like form. [This scroll is generated by a line meeting the quartic and a line through the tacnode and further determined by the condition of lying in a plane which has a (1, 1) correspondence with the point on the director line through which the generators lying in it pass. The director line is a tacnodal line. If the quartic have an additional node we get the next case (XIII) in which there is a

double generator.] This will be the case if any factor $y - ax$ of u_2 enters twice into u_4. In that case it is obvious that the line $y - ax$, $aw - z$ is a double line on the surface. This is Cayley's fifth species.

Every quartic scroll may be classed under one of the species which we have enumerated.

[554*a*. *Steiner's Quartic.*—It was shown (Art. 523, Ex. 2) that the reciprocal of a four-nodal cubic surface is a quartic with three nodal lines meeting in a point. Conversely, every such quartic is the reciprocal of such a cubic, for by proper choice of the tetrahedron of reference and the implied constants the equation of the quartic may be taken as

$$S \equiv y^2z^2 + z^2x^2 + x^2y^2 - 2xyzw = 0.$$

The coordinates of any point on the surface are then

$$x : y : z : w :: 2\beta\gamma : 2\gamma\alpha : 2\alpha\beta : \alpha^2 + \beta^2 + \gamma^2,$$

α, β, γ being parameters which may be considered as the trilinear coordinates of a point on a plane whose points thus have a (1, 1) correspondence with the points on S.

To the section of S by the plane $\lambda x + \mu y + \nu z + \rho w$ will correspond the conics of the family

$$C \equiv \rho(\alpha^2 + \beta^2 + \gamma^2) + 2\lambda\beta\gamma + 2\mu\gamma\alpha + 2\nu\alpha\beta = 0,$$

to the tangent planes of S will correspond the line-pairs of C, and to coincident line-pairs of C will correspond tropes, i.e. planes touching S all along a conic.

Now the members of C which are perfect squares are $(\alpha \pm \beta \pm \gamma)^2$, so that S has four tropes, namely,

$$x + y + z + w = 0 \qquad x - y - z + w = 0$$
$$y - z - x + w = 0 \qquad z - x - y + w = 0,$$

and hence the reciprocal of S (which is clearly of the third degree as is seen by forming the discriminant of C) has four conic nodes.

If the tropes be taken for planes of reference the equation of the surface takes the form given in the note to Ex. 2, Art. 523.

There are two special cases of Steiner's quartic arising from the coincidence of a pair or of all three nodal lines.

The equation of the surface when the nodal lines are xy counted twice and xz can be obtained from the general form

$$ay^2z^2 + bz^2x^2 + cx^2y^2 + dxyzw = 0$$

by putting $x + \lambda z$ for z in this and then conceiving λ to tend to zero and a and c to infinity in such a way that $a + c$ tends to zero and $a\lambda$ remains finite.

We thus obtain an equation of the form

$$x^4 + y^2z^2 + x^2yw = 0,$$

which shows that the line xy is a tacnodal line, and two of the tropes coincide with the plane y which touches all along the tacnodal line.

If the three nodal lines coincide they become an *oscnodal* line, that is, one such that any plane section of the quartic has an oscnode when its plane meets the line. Taking $y^2 - zx$ as the cone which osculates the surface all along this line (xy), and also passes through the conic of contact of the solitary trope w which has not coincided with the tangent plane along xy, we see that the equation of the surface must be

$$(y^2 - zx)^2 = x^3w,$$

a form which can be verified by the method of the examples below.

Steiner's quartic is the only surface other than quadrics and cubic scrolls which contains an infinity of conics passing through any point on it. This was shown by Darboux and generalised by Castelnuovo (*Rend. Lincei*, 1894), who states that *outside ruled surfaces Steiner's quartic is the only surface which is cut in non-proper curves by all planes of a doubly infinite series.*

Ex. 1.* The locus of a point whose coordinates are proportional to quadric functions of three homogeneous parameters is a Steiner quartic.

From what proceeds it will be sufficient to show that in the family of conics

$$\lambda U_1 + \mu U_2 + \nu U_3 + \rho U_4 = 0,$$

where $U_r \equiv (a_r b_r c_r f_r g_r h_r)(\alpha\ \beta\ \gamma)^2$, there are four members which reduce to a pair of coincident lines.

This is so because a linear tangential system of conics $\Sigma + \kappa\Sigma'$ can be determined such that the invariant Θ vanishes between any member of this

* These examples are chiefly from the papers cited Art. 545.

system and the former. Now it is easy to see that this relation is satisfied for a conic and the square of the equation of any of its tangents; consequently the four common tangents of the second system each counted twice must be members of the first.

Ex. 2. If the system $\Sigma + \kappa\Sigma'$ of Ex. 1 has two coincident common tangents, we get the second kind of Steiner's quartic. Let $\Sigma = \eta\zeta$, $\Sigma' = \xi(\eta + \zeta)$.

The conics of the first system may be taken to be then

$$\lambda\alpha^2 + \mu\beta^2 + \nu\gamma^2 + \rho\alpha(\beta - \gamma) = 0,$$

whence $w = \sqrt{x}\,(\sqrt{y} - \sqrt{z})$, an equation which, when rationalised, is equivalent to that already given.

Ex. 3. If the system $\Sigma + \kappa\Sigma'$ has three common tangents, the same method gives rise to the third kind of Steiner quartic. Here proceeding similarly we find $x : y : z : w :: \beta^2 : \gamma^2 : \alpha\gamma : \alpha^2 - \beta\gamma$ whence $(yw - z^2)^2 = xy^3$.

Ex. 4. The second and third kind of Steiner's surface are reciprocal to cubic surfaces with the singularities $2C_2 + B_4$ and $C_2 + B_6$ respectively. This is seen by forming the discriminant of the system $\lambda U_1 + \mu U_2 + \nu U_3 + \rho U_4$.

Ex. 5. The locus of poles of a given plane with respect to conics on Steiner's surface is another Steiner's surface. (Montesano.)]

555. The only quartics with nodal lines which have not been considered are those which have a nodal right line or a nodal conic. In either case the surface contains a finite number of right lines. For take an arbitrary point on the nodal line, and an arbitrary point on any plane section of the surface, and the line joining them will only meet the surface in one other point. We can, by Joachimsthal's method, obtain a simple equation determining the coordinates of that point in terms of the coordinates of the extreme points. In order that the line should lie altogether on the surface, both members of this equation must vanish; that is to say, two conditions must be fulfilled. And since we have two parameters at our disposal we can satisfy the two conditions in a finite number of ways.* In the case where the quartic has a nodal right line xy, substituting $y = \lambda x$ in the equation, and

* The same argument proves that if a surface of the n^{th} order have a multiple line of the $(n - 2)^{\text{th}}$ order of multiplicity, the surface will contain right lines. If the multiple line be a right line it is easily proved, as in Art. 530, that the number of other right lines is $2(3n - 4)$. If the multiple line be not plane, or if the surface possess in addition any other multiple line, the surface is generally a scroll. See a paper by R. Sturm, *Math. Annalen*, t. IV, (1871).

proceeding, as in Art. 530, we find that eight planes can be drawn through the nodal line which meet the surface, each in two other right lines, and thus that there are sixteen right lines on the surface besides the nodal line.

[Ex. 1. The section of the surface in this case by any triple tangent plane must be a pair of conics intersecting on the nodal line and at the points of contact.

Ex. 2. These conics are intersected, the one by one, and the other by the second of the two right lines lying in any of the eight planes of this Article.

(If both lines met the same conic the intersection of their plane and the triple tangent plane would meet that conic in three points.)]

[555*a*. *Birational Transformation into a Plane.*—The general theory of this process is treated in the next chapter, but it is easy to see from the facts just established that a one-one correspondence can be set up between the points in an arbitrary plane and the points on a quartic surface with a nodal line.

When this is a right line (D) we have seen that there are conics lying on the surface which intersect D.

Let P be an arbitrary point in a fixed plane. Through P one line can be drawn meeting D and a certain specified one of these conics, and this meets the surface again in *one* point Q which is uniquely determined by P, and conversely, Q determines P. When D is a conic we have seen that there are right lines meeting it lying on the surface, and an exactly equivalent process is possible, the only difference being that here the conic is double and the line single.

If the nodal line is of higher order than unity, and is not plane, the surface is a scroll. It will be found that similar reasoning is valid in the case of all these except those numbered XI and XII.

These two and quartics having isolated nodes do not permit of the transformation in question.]

556. We do not attempt to give a complete account of the different kinds of nodal lines on a quartic, the varieties being very numerous, but merely indicate some of the cases which

would need to be considered in a complete enumeration.* The general equation of a quartic with a nodal right line may be written

$$u_4 + zu_3 + wv_3 + z^2t_2 + zwu_2 + w^2v_2 = 0,$$

where u_4, u_3, &c., are functions in x and y of the order indicated by the suffixes. Now, attending merely to the varieties in the last three terms, and numbering the general case (1), we have the following additional cases; (2) the three quantities t_2, u_2, v_2 may have a common factor. In this case one of the tangent planes is the same along the double line, and one of the sixteen lines on the surface coincides with that line; (3) the last terms may be divisible by a factor not containing x or y, and so be reducible to the form $(az + bw)(zu_2 + wv_2)$; (4) there may be both a factor in x and y and also in z and w, the terms being reducible to the form $(ax + by)(a'z + b'w)(xz + yw)$; (5) we may have t_2, u_2, v_2 only differing by numerical factors, in which case there are two fixed tangent planes along the double line, and the case may be distinguished when the factor in z and w is a perfect square, that is to say, we have the two cases: (5a) the terms of the second degree reducible to the form $xyzw$, and (5b) reducible to the form xyz^2; (6) the three terms may break up into the factors $(xz - yw)(zu_1 + wv_1)$; (7) the terms may form a perfect square $(xz + yw)^2$, in which case the line is cuspidal, the two tangent planes at each point coinciding but varying from point to point; (8) the cuspidal tangent plane may be the same for every point; the three terms being reducible to the form (8a), x^2zw, or (8b), x^2z^2. This enumeration does not completely exhaust the varieties; and we have not taken into consideration the varieties resulting from taking into account the preceding terms, as for instance, if a factor $xz + yw$ divide not only the last three terms but also the terms $zu_3 + wv_3$. From the theory of reciprocal surfaces afterwards to be given, it appears that a quartic with an ordinary double line is of the twentieth class, and that when the line is cuspidal the class reduces to the twelfth. It would need to

* On the subject of multiple right lines on a surface, the reader may consult a memoir by Zeuthen, *Math. Annalen*, IV. (1871).

be examined whether the class might not have intermediate values for special forms of the double line, and, again, what forms of the double line intervene between the cuspidal and the tacnodal for which we have seen that the surface is a scroll, the class being the fourth. [In the general case there are four pinch-points; in (2) these have coincided in pairs; in (3) there are two pinch-points and a triple point, while in (4) the pinch-points have coincided; finally in (5*a*) and (5*b*) we have two distinct or coincident triple points.

On cuspidal lines there are singularities called *tacnodal* points, that is, points at which any section has a tacnode instead of a cusp. For these and other details of special nodal lines, Basset's *Surfaces* (Arts. 255-264, and Chapter V generally) should be consulted.]

557. A quartic with a nodal line may have also double points. Two of the eight planes which meet the surface in right lines will coincide with the plane joining the nodal line to one of the nodal points. It is easy to write down the equation of a quartic with a nodal line and four nodal points. For let U, V, W represent three quadrics having a right line common and consequently four common points, then any quadratic function of U, V, W represents a quartic on which the line and points are nodal.

There are in the case just mentioned four planes, each passing through the nodal line and a nodal point, each such plane meeting the surface in the nodal line twice, and in two lines intersecting in the nodal point. There are at most four planes containing a nodal point, but any such plane may meet the surface in the nodal line twice, and in a two-fold line having upon it *two* nodal points; the surface may thus have as many as eight nodal points. The quartic with eight nodes and a nodal line is Plücker's *Complex Surface* (Art. 455), and its equation is

$$\begin{vmatrix} & x, & y, & 1 \\ x, & a, & h, & g \\ y, & h, & b, & f \\ 1, & g, & f, & c \end{vmatrix} = 0,$$

where a, b, h are of form $(z, w)^2$; f, g of form $(z, w)^1$, and c is constant. There are through the nodal line four planes, the section by each of them being a two-fold line, and on each such two-fold line there are two nodes. [For the plane $w = \theta z$ meets the surface in zw counted twice and in a conic whose tangential equation is $(a'b'c'f'g'h')(\lambda\mu\nu)^2 = 0$, where for instance a' means the result of putting $\theta z =$ for w in a and dividing by z^2. The discriminant of this is of the fourth order in θ, hence there are four planes for which this equation represents a point-pair. These are double points, for every line through them in the corresponding planes must be considered a tangent line to the surface. The section by these planes is given by the point-equation corresponding to the point-pair and is accordingly the line joining them counted twice.]

Suppose that the pairs of nodes are 1, 2; 3, 4; 5, 6; 7, 8; so that 12, 34, 56, 78 each meet the nodal line. For a node 1, the circumscribed sextic cone is $P^2 U_4 = 0$, where P is the plane through the double line—this should contain the lines 12, 13, 14, 15, 16, 17, 18 each twice; but P contains the line 12, and therefore P^2 contains it twice; hence, U_4 should contain the remaining six lines each twice, that is, it breaks up into four planes $ABCD$ which intersect in pairs in the six lines. Taking in like manner $P'^2 A'B'C'D' = 0$ for the sextic cone belonging to the node 2, the eight nodes lie by fours in the eight planes A, B, C, D, A', B', C', D', and through each of the nodes there pass four of these planes; it is easy to construct geometrically such a system of eight points lying by fours in eight planes; the figure may be conceived of as a cube divested of part of its symmetry.

A special case would arise if one or more of the nodal points were to coincide with the nodal line. Thus the equation

$$\begin{aligned} ax^4 + bx^3y + cx^2y^2 + dxy^2(y - mw) + ey^2(y - mw)^2 + (Ax^3 + Bx^2y \\ + Cxy^2)z \\ + Dy^2z(y - mw) + (A'x^3 + B'x^2y)w + C'xyw(y - mw) \\ + (\alpha x^2 + \beta xy + \gamma y^2)z^2 + (\alpha' x^2 + \beta' xy)zw + \alpha'' x^2w^2 = 0, \end{aligned}$$

represents a quartic having the line xy as nodal and the point $x, z, y - mw$ as a nodal point; and if in the above we make $m = 0$, the point will lie on xy. The kind of nodal line here indicated appears to be different from any of those previously considered.

[Ex. 1. The equation of Plücker's Complex Surface can be written

$$\sqrt{(m-n)xX} + \sqrt{(n-l)yY} + \sqrt{(l-m)zZ} = 0,$$

where
$$X \equiv u'y - t'z + p'w = 0$$
is the plane joining $(1, 0, 0, 0)$ to the nodal line (p', q', r', s', t', u') with similar meanings for Y and Z.

It is easy to verify that this is the surface just discussed. The following proof starts from the definition given in Art. 455 and thus establishes the identity of the surface of the present article with the general Plücker's Complex Surface.

Let $\Phi = 0$ be the given quadratic complex and p', q', etc., the line coordinates of the given line. It will make no difference if instead of Φ we take the complex

$$\Phi + (Ap + Bq + Cr + Ds + Et + Fu)\,(s'p + t'q + r'u + p's + q't + r'u) = 0.$$

By a proper choice of the planes of reference, we can reduce the left-hand member of this equation to the form $lps + mqr + nst$, for we have at our disposal the six quantities A, B, etc., and the twelve implied in the planes of reference to satisfy the eighteen equations obtained by equating to zero the coefficients of the remaining terms. Writing then

$$\Phi \equiv lps + mqr + nst = 0,$$
$$ps + qr + st = 0,$$

we have
$$ps : qr : st :: m - n : n - l : l - m.$$

The lines of Φ which lie in $ax + by + cz + dw = 0$, satisfy the equation $as + bt + cu = 0$ and therefore satisfy

$$\frac{a(m-n)}{p} + \frac{b(n-l)}{q} + \frac{c(l-m)}{r} = 0,$$

and hence lie in tangent planes to the cone

$$\sqrt{a(m-n)x} + \sqrt{b(n-l)y} + \sqrt{c(l-m)z} = 0.$$

Plücker's surface is accordingly the locus of the conic in which this cone meets the plane $ax + by + cz + dw = 0$. Solve for the ratios $a : b : c$ between the equation of the plane and any two of the relations expressing that it passes through the given line (Art. 57b) and substitute in the equation of the cone, when the required equation of the surface is obtained. It may be noted that this is one of four similar forms, another being

$$\sqrt{(l-m)yY} + \sqrt{(n-l)zZ} + \sqrt{(m-n)wW} = 0.$$

Ex. 2. The planes X, Y, Z, W are those for which the complex conic reduces to a pair of points, and they touch the surface all along the joining line. Reciprocally the four points when the nodal line meets the coordinate planes are those for which the complex cone is a pair of planes. Every plane

through one of these points has it as a cusp on the curve in which the plane meets the surface.

Ex. 3. The locus of poles of the nodal line with respect to the complex conics lying in planes through it is a right line.]

QUARTICS WITH NODAL CONICS—CYCLIDES.

[558. We come now to quartics with a nodal conic, including the case where the conic breaks up into a pair of right lines. Segre (*Math. Ann.*, XXIV.) first made a complete enumeration of the different species of this class of quartic. His method of classification depends on considering such surfaces as the "projections" on to a three-dimensional space of the "surface" common to two quadratic "varieties" in four-dimensional space.

He obtains seventy-six different species, but this number includes several surfaces which have already been considered, namely, the scrolls VIII, IX, X, XIII, and the subform of X; the three kinds of Steiner's quartic; and thirteen included in the classification of Art. 556. Rejecting these there remain fifty-five thus divided: sixteen with a proper nodal conic; seven with a proper cuspidal conic; eighteen with two nodal lines not intersecting in a triple point, and fourteen when they intersect in a triple point. The subforms arise from the isolated nodes that may exist in addition and from one or both of the nodal right lines becoming cuspidal or torsal. The theory of the ordinary case of two intersecting nodal right lines is included in that when the nodal line is a proper conic.]

559. In this case any arbitrary plane meets the surface in a binodal quartic; if the plane be a tangent plane the quartic will be trinodal; if the plane be doubly a tangent plane the quartic will break up into two conics.* If the plane touch three times, the section must have an additional double point; that is to say, one of the conics must break up into two right lines; and since a surface has in general a definite number

* It was from this point of view these surfaces were studied by Kummer, viz. as quartics on which lie an infinity of conics.

of triple tangent planes we see, as we have already inferred from other considerations, that the surface contains a definite number of right lines. This number is sixteen, as may be shown by the method indicated, Art. 555, but we do not delay on the details of the proof, as we shall have occasion afterwards to show how the theorem was originally inferred by Clebsch. Each of the sixteen lines is met by five others, the relation between the lines being connected by Geiser and Darboux with the twenty-seven lines of a cubic surface as follows: If on a cubic surface we disregard any one line and the ten lines which meet it, then the sixteen remaining lines are, in regard to their mutual intersections, related to each other as the sixteen lines on the quartic.

In fact this is easily shown by the method of inversion in the case where the nodal conic is the circle at infinity, a case to which the general form can always be reduced by homographic transformation. [The equation of such a quartic can be written (see Art. 562)

$$ax^2 + by^2 + cz^2 + 2lx + 2my + 2nz + (x^2 + y^2 + z^2)^2 = 0.$$

The inverse of this is the circular cubic

$$1 + ax^2 + by^2 + cz^2 + 2(lx + my + nz)(x^2 + y^2 + z^2) = 0.$$

Now a right line passing through the circle at infinity inverts into a right line also passing through the circle at infinity. Any lines that exist on the quartic meet this circle and therefore invert into lines on the cubic which meet the circle.] Of the twenty-seven right lines on this cubic, one lies in the plane at infinity, ten meet that line, and the remaining sixteen meet the circle at infinity; and these last, and these only, are inverted into right lines on the quartic.

The lines may be grouped in "double fours," such that in a double four each line of the one four meets three lines of the other four; but no two lines of the same four meet each other. There are in all twenty double fours, each line therefore entering into ten of them. [This theorem follows from the notation for the lines explained in the last chapter. If the eleven lines we omit are c_{56} and the ten which meet it, the twenty "double fours" are

$$\begin{Bmatrix} a_1a_2a_3a_4 \\ b_1b_2b_3b_4 \end{Bmatrix} \text{ and } \begin{Bmatrix} c_{15}c_{25}c_{35}c_{45} \\ c_{16}c_{26}c_{36}c_{46} \end{Bmatrix}$$

and six each of the following types

$$\begin{Bmatrix} a_1a_2c_{45}c_{35} \\ c_{26}c_{16}b_3b_4 \end{Bmatrix} ; \begin{Bmatrix} a_1a_2c_{46}c_{36} \\ c_{25}c_{15}b_3b_4 \end{Bmatrix} ; \begin{Bmatrix} a_1b_1c_{25}c_{26} \\ a_2b_2c_{15}c_{16} \end{Bmatrix}.$$

There are altogether 120 tetrads of lines such that no two of a tetrad intersect. Forty of these are those just mentioned; the remaining eighty have the property that there is one and only one line of the remaining twelve lines which fails to meet any of the tetrad. For instance, $a_1a_2a_3c_{45}$ is not met by c_{46}. It is easy to see that each line meets five others. The plane of two lines is a triple tangent plane and therefore there are $\frac{1}{2}(16 \times 5)$ or forty such planes. In the examples which follow some of these theorems are proved independently of the theory of the cubic surface.

Ex. 1. Every line on a quartic having a nodal conic meets five other lines on the surface. Let $w = 0$, $Fyz + Gzx + Hxy = 0$ be the conic and yz the line, so that the quartic is

$$by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2myw + 2nzw + \{Fyz + Gzx + Hxy\}\{Fyz + Gzx + Hxy + w(px + qy + rz)\} = 0.$$

Put $y = \mu z$ in this and write down the conditions that the result should permit of the factor

$$\theta w + \mu Fz + (G + \mu H)x.$$

It will be found that θ must satisfy two equations of the form

$$\theta^3 + \kappa_1\theta^2 + \kappa_2\theta + \lambda_1\mu_1 = 0$$
$$\theta^2 + \nu_1\theta + \lambda_1\rho_1 = 0$$

where the suffixes denote the degree of the coefficients in μ. The eliminant of these two equations is of the 6th degree in μ, but contains the factor λ_1 which the work shows to be irrelevant.

Ex. 2. Infer the existence of sixteen lines from the fact that each line meets five others.

Let 2, 3, 4, 5, 6 all meet 1. A unique quadric can be described through 1, 2, 3 and the nodal conic, and its intersection with the quartic consists of the conic twice and 1, 2, 3, and therefore of one other line—say (23)—which meets 2 and 3 since it cannot meet 1. We get ten lines of the type (23), making sixteen in all. Further (23) meets (45) because the quartic and the two quadrics determining these lines must have one point of intersection not absorbed by the common curve, the line 1 and the conic (see Art. 355). It will now be found that of the set of sixteen lines each is met by five of the set.

Ex. 3.* The transformation $x : y : z : w :: X^2 : XY : XZ : YW - Z^2$ transforms a cubic surface though $y = 0$, $xw + z^2 = 0$ into a quartic having $X = 0$,

* See Geiser, *Crelle*, LXX., and Cremona, *Rend. Inst. Lomb.*, 1871.

$YW - Z^2 = 0$ for a nodal conic. The sixteen lines of the cubic meeting the conic become lines on the quartic; the line of the cubic in the plane y becomes the point XYZ; the other ten lines become conics touching Y at XYZ. (The cubic surface is assumed not to touch x at xyz.)]

[559*a*. It is shown as in Art. 562 that there are five quadric cones (known as Kummer's cones after their discoverer) all of whose edges are bitangent to the surface. Consider a point where one of the sixteen lines meets one of these cones. The tangent plane to the quartic must contain the line, but it is also the tangent plane to the cone. Hence *all the lines touch all the cones.* We can now prove a very interesting theorem due to Zeuthen:* *The cone of contact to the quartic from a point on the nodal conic is a non-singular quartic cone whose bitangent planes are* (*a*) *the sixteen planes joining the vertex to the lines,* (*b*) *the ten tangent planes to the cones from the vertex,* (*c*) *the nodal tangent planes to the surface at the vertex.*

In fact the sixteen planes are of course tangent to the surface at two points on the corresponding line; every tangent plane of a Kummer cone is a bitangent plane; and finally the tangent plane of a surface is always a bitangent plane of the tangent cone from its point of contact, namely, along the two inflexional tangents, and so when there is a nodal line the tangent planes to both sheets are bitangent planes.

Since the existence of twenty-eight bitangent planes has been demonstrated, it further follows that the cone is non-singular.

If the vertex is taken at a point where one of the lines L meets the nodal conic, L is a nodal edge of the cone, for every plane through it touches the quartic twice. The cones have now only sixteen proper bitangents, namely, those five tangent planes to Kummer's cones which do not contain L, the ten planes to the ten lines which do not meet L, and the tangent plane to that sheet of the quartic which does not contain L. The tangent plane to the other sheet, and the planes

* *Ann. di Mat.*, XIV. p. 34.

joining L to the five lines which meet it are the six tangent planes to the cone through the nodal line.]

560. In what follows, we suppose the surface to be a cyclide, as the term is used by Casey and Darboux, that is to say, having the circle at infinity as the nodal conic: and in order to generalise the results, it is only necessary in the equations of the nodal line $w=0$, $x^2+y^2+z^2=0$, to suppose x, y, z, w to be any four planes; while in the special case w is at infinity, and x, y, z are ordinary rectangular coordinates. The properties of the cyclide may be studied in exactly the same manner as the properties of bicircular quartics were treated (*Higher Plane Curves*, Arts. 251, 272, &c.). Consider any quartic whose equation may be written $(X, Y, Z, W)^2=0$, where X, Y, Z, W represent quadrics, and we equate to zero a complete quadratic function of these quantities. By a linear transformation of these quantities we may reduce this equation as the general equation of the second degree was reduced, and so bring it to either of the forms $aX^2+bY^2+cZ^2+dW^2=0$, or $XY=ZW$,* only in the latter case the separate factors are not necessarily real. From the latter form it is apparent that there are on such a quartic at least two singly infinite series of quadriquadric curves, and that through two curves belonging one to each system can be drawn a quadric $\lambda\mu X-\lambda Z-\mu W+Y=0$, touching the surface in the eight points where these curves intersect. And, generally, the quadric $aX+\beta Y+\gamma Z+\delta W$ will touch the quartic, provided a, β, γ, δ satisfy the familiar relation of Art. 79. All quadrics included in this form have a common Jacobian on which will lie all possible vertics of cones involved in the system. Thus,

* It has been shown by Valentiner, *Zeuthen Tidsskrift* (4), III., that the form of the equation of a quartic here considered is not of the greatest generality, and in fact that any surface of the n^{th} degree which contains the complete curve of intersection of two surfaces must be a special surface when n exceeds 3. The equation of a quartic which contains a quadriquadric curve depends on only 33 independent constants. [See as well Sisam (*l.c.* Art. 545) who shows also that a quadric function of *five* quadrics *is* sufficiently general to represent any quartic.]

through each of the quadriquadric curves just spoken of, can be drawn four cones whose vertices lie on the Jacobian.

A special case is when the equation of the quartic can be expressed in terms of three quadrics only $(X, Y, Z)^2 = 0$. This cannot happen unless the quartic have double points, since all points common to the three quadrics X, Y, Z are double points on the quartic. In this case the equation can be brought by linear transformation to either of the forms

$$aX^2 + bY^2 + cZ^2 = 0, \text{ or } XZ = Y^2.$$

Such a quartic is evidently the locus of the system of curves $Y = \lambda X$, $Z = \lambda Y$, and the quadric $\lambda^2 X - 2\lambda Y + Z$ touches the quartic along the whole length of this curve. The generators of any quadric of this system are bitangents to the quartic.

561. To apply this to the cyclide, it is easy to see that if X, Y, Z, W be four spheres, the equation $(X, Y, Z, W)^2 = 0$ is general enough to represent any cyclide. Since the Jacobian of four spheres is the sphere which cuts them at right angles, all spheres of the system $\alpha X + \beta Y + \gamma Z + \delta W$ cut a fixed sphere orthogonally. Further, the coordinates of the centre of any such sphere are easily seen to be proportional to linear functions of α, β, γ, δ; and, reciprocally, these quantities are proportional to linear functions of these coordinates. Thus the condition of contact (Art. 79) being of the second degree in α, β, γ, δ, establishes a relation of the second degree in these coordinates. Hence we have a mode of generation for cyclides corresponding to that given for bicircular quartics (*Higher Plane Curves*, Art. 273), viz. a cyclide is the envelope of a sphere whose centre moves on a fixed quadric F, and which cuts a fixed sphere J orthogonally. From this mode of generation several consequences immediately follow. First, the cyclide is its own inverse with regard to the sphere J; for any sphere which cuts J orthogonally is its own inverse in respect to it, so that the generating sphere not being changed by inversion, neither is the envelope. Thus, the cyclide is an anallagmatic surface, see note, Art. 515. Secondly, the intersection of F and J is a focal curve of the cyclide; for the

Jacobian J is the locus of all point-spheres belonging to the system $\alpha X + \beta Y + \gamma Z + \delta W$; and therefore, from the mode of generation, every point of the curve FJ is a point-sphere having double contact with the quartic; that is to say, is a focus. Thirdly, in the case where the centre of the enveloped sphere is at infinity on F, the sphere reduces to a plane through the centre of J (or more strictly to that plane, together with the plane infinity). It follows then, that if a cone be drawn through the centre of J whose tangent planes are perpendicular to the edges of the asymptotic cone of F, these tangent planes are double tangent planes to the quartic, which they meet therefore each in two circles, while the edges of this cone are bitangent lines to the quartic.

[561*a*. The following is a direct elementary proof of the main properties of cyclides, starting from their definition as the envelope of a sphere whose centre moves on a quadric F and which cuts a fixed sphere J orthogonally.

By the ordinary rules for finding envelopes the equation of the cyclide is found to be

$$S \equiv [x^2 + y^2 + z^2 + r^2 - \alpha^2 - \beta^2 - \gamma^2]^2 - 4[a^2(x-\alpha)^2 + b^2(y-\beta)^2 + c^2(z-\gamma)^2] = 0$$

$$\text{if } F \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \text{ and } J \equiv (x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2.$$

Let us enquire if other values a', b', c', α', β', γ', r' could be given to the constants in S and yet leave its equation unaltered.

This requires

$$a'^2 - a^2 = b'^2 - b^2 = c'^2 - c^2 = \lambda \text{ say} \quad . \quad . \quad . \quad . \quad \text{(i)}$$

$$r'^2 - \alpha'^2 - \beta'^2 - \gamma'^2 = r^2 - \alpha^2 - \beta^2 - \gamma^2 \quad . \quad . \quad . \quad \text{(ii)}$$

$$\alpha'(a^2+\lambda) - \alpha a^2 = \beta'(b^2+\lambda) - \beta b^2 = \gamma'(c^2+\lambda) - \gamma c^2 = 0 \quad . \quad \text{(iii)}$$

$$\lambda\left\{\frac{\alpha^2}{a^2+\lambda} + \frac{\beta^2}{b^2+\lambda} + \frac{\gamma^2}{c^2+\lambda} - 1 - \frac{r^2}{\lambda}\right\} = 0 \quad . \quad . \quad \text{(iv)}$$

The last equation shows that in general four other ways of generating S exist; (i) shows that the four new quadrics will be confocal with F, and from the last three equations it is easy to show that each of the four new spheres are orthogonal to J and each other.

The following special cases may be noted :—

1. J is a point-sphere and hence two roots of (iv) are zero, and the point lies on the other three quadrics.

2. J touches F, and two roots of (iv) are equal by Art. 202.

3. J is a point-sphere on F, and three roots of (iv) are zero, and the point lies on the other two quadrics.

4. J and F have stationary contact, and three roots of (iv) are equal by Art. 206.

The first two cases represent of course the same kind of cyclide, and so do the last two, the only difference being the system from which we start. The point J in case 1 is easily seen to be a conic node and in case 3 a binode. Hence we see that *cyclides with a node have only four systems of enveloping spheres, and one of the J spheres is a point-sphere at the node lying on the other three F quadrics*, and that a corresponding theorem is true for cyclides having a binode obtained by changing "*four*" to "*three*" and "*three*" to "*two*".]

562. We have thus far considered the equation of the cyclide expressed in terms of four quadrics; but it is even more obvious that the equation can be expressed in terms of three quadrics. In fact, the equation of a quadric having for nodal line the intersection of the quadric U by the plane P, may obviously be written $U^2 = P^2 V$. Or, again, if we write down the following most general equation of a quartic, having as a nodal line the intersection of $x^2+y^2+z^2$, and w,

$$(x^2+y^2+z^2)^2 + 2wu_1(x^2+y^2+z^2) + w^2u_2 = 0\,;$$

this can obviously at once be written in the above form as,

$$(x^2+y^2+z^2+wu_1)^2 = w^2v_2.$$

We can simplify this equation by transformation to parallel axes through a new origin, so as to make the u_1 disappear, and we may suppose the axes of coordinates to be parallel to the axes of the quadric v_2, so that v_2 does not contain the terms yz, zx, xy. It appears then, from what has been said, that the cyclide, the general equation being reduced to the form

$$(x^2+y^2+z^2)^2 = ax^2+by^2+cz^2+2lx+2my+2nz+d = V,$$

is the envelope of the quadric $V + 2\lambda(x^2+y^2+z^2)+\lambda^2 = 0$, every quadric of this system touching the quartic at every point where it meets it. The discriminant of this quadric equated to zero gives

$$\frac{l^2}{a+2\lambda}+\frac{m^2}{b+2\lambda}+\frac{n^2}{c+2\lambda} = d+\lambda^2,$$

and this equation being a quintic in λ, we see that there are five values of λ for which this quadric reduces to a cone, and therefore five cones whose edges are bitangents to the quartic. Taking this in connection with what was stated at the end

of the last article, it may be inferred that there are five spheres J, each of which combined with a corresponding quadric F gives a mode of generating the cyclide. And this may be shown directly by investigating the condition that the sphere $x^2+y^2+z^2-u_1$ should have double contact with the cyclide, or meet it in two circles. For, substituting in the equation of the cyclide we get $u_1^2-V=0$, and if we subtract from this $2\lambda(x^2+y^2+z^2-u_1)$ and determine λ by the condition that the difference shall represent two planes, we get the same quintic as before for λ; and we find also that the centre of the sphere must satisfy the equation

$$\frac{a^2}{a+2\lambda}+\frac{\beta^2}{b+2\lambda}+\frac{\gamma^2}{c+2\lambda}+\tfrac{1}{4}=0,$$

from which we see that there are five series of double tangent spheres; that the locus of the centre of the spheres of each series is a quadric, and that the five quadrics are confocal.

It appears from what has been said that through any point can be drawn ten planes cutting the cyclide in circles, namely, the pairs of tangent planes which can be drawn through the point to the five cones.*

563. The five-fold generation may be shown in another way. If we suppose the quadric locus of centres F to be identical with the sphere J which is cut orthogonally, we evidently get for the cyclide J itself counted twice. Again, if we have two cyclides both expressed in the form $(X, Y, Z, W)^2=0$, it appears from the theory of quadrics that by substituting for X, Y, Z, W linear functions of these quantities both can be expressed in the form $aX^2+bY^2+cZ^2+dW^2$. Thus then it is possible to express the equation of any cyclide in the form $a'X^2+b'Y^2+c'Z^2+d'W^2$, while at the same time we have an identical equation $J^2=aX^2+bY^2+cZ^2+dW^2$. For the actual transformation we refer to Casey, p. 599, Darboux, p. 135, but we can show in another way what this identical

* [The details of the work will be found in a paper by J. Fraser, *Proc. R.I.A.*, XXIV., A, No. 10, in which also the actual reduction to the form of Art. 563 is effected.]

equation is. Multiply by the ordinary rule the two determinants

$$\begin{vmatrix} \rho^2 & -x, & -y, & -z, & 1 \\ d, & -l, & -m, & -n, & 1 \\ d', & -l', & -m', & -n', & 1 \\ d'', & -l'', & -m'', & -n'', & 1 \\ d''', & -l''', & -m''', & -n''', & 1 \end{vmatrix} \begin{vmatrix} 1, & 2x, & 2y, & 2z, & \rho^2 \\ 1, & 2l, & 2m, & 2n, & d \\ 1, & 2l', & 2m', & 2n', & d' \\ 1, & 2l'', & 2m'', & 2n'', & d'' \\ 1, & 2l''', & 2m''', & 2n''', & d''' \end{vmatrix},$$

(where we have written for brevity ρ^2 instead of $x^2+y^2+z^2$, and where either determinant equated to zero gives the equation of the sphere cutting orthogonally four spheres), and the product is

$$\begin{vmatrix} 0, & X, & Y, & Z, & W \\ X, & -2r^2, & (12), & (13), & (14) \\ Y, & (12), & -2r'^2, & (23), & (24) \\ Z, & (13), & (23), & -2r''^2, & (34) \\ W, & (14), & (24), & (34), & -2r'''^2 \end{vmatrix},$$

where (12) is $d+d'-2ll'-2mm'-2nn'$, and vanishes if the two spheres cut each other orthogonally. On the supposition then that each pair of the four given spheres cut orthogonally, the square of the equation of the sphere cutting them at right angles is proportional to

$$\begin{vmatrix} 0, & X, & Y, & Z, & W \\ X, & -2r^2, & 0, & 0, & 0 \\ Y, & 0, & -2r'^2, & 0, & 0 \\ Z, & 0, & 0, & -2r''^2, & 0 \\ W, & 0, & 0, & 0, & -2r'''^2 \end{vmatrix},$$

whence it immediately follows that if five spheres cut each other orthogonally, the identical relation subsists

$$\frac{X^2}{r^2}+\frac{Y^2}{r'^2}+\frac{Z^2}{r''^2}+\frac{V^2}{r'''^2}+\frac{W^2}{r^{iv2}}=0.$$

[To complete the argument it is necessary to show conversely that if a linear relation connects the squares of five spheres they are mutually orthogonal. This is shown at the end of Art. 566. J, X, Y, Z, W being now proved orthogonal, inversion of the cyclide with respect to any of them obviously leaves its equation unaltered.]

It may be noted in passing, that in virtue of this identity, the equation $W=0$ may be written in the form

$$\left(\frac{X-W}{r}\right)^2+\left(\frac{Y-W}{r'}\right)^2+\left(\frac{Z-W}{r''}\right)^2+\left(\frac{V-W}{r'''}\right)^2=0,$$

showing that the sphere W meets the four others in four planes, which form a self-conjugate tetrahedron with respect to W. To return to the cyclide, it having been proved that its equation may be written in the form

$$aX^2 + bY^2 + cZ^2 + lV^2 = 0,$$

and that it may be generated as the envelope of a sphere cutting W orthogonally, we may, by the help of the identity just given, eliminate any other of the quantities X, Y, &c., and write for example the equation in the form

$$a'Y^2 + b'Z^2 + c'V^2 + d'W^2 = 0,$$

and generate the cyclide as the envelope of a sphere cutting X orthogonally.

564. The condition that two surfaces whose equations are expressed in terms of the five spheres X, Y, Z, V, W should cut each other orthogonally, admits of being simply expressed. It is in the first instance

$$\left(\frac{d\phi}{dX}\frac{dX}{dx} + \text{\&c.}\right)\left(\frac{d\psi}{dX}\frac{dX}{dx} + \text{\&c.}\right)$$
$$+ \left(\frac{d\phi}{dX}\frac{dX}{dy} + \text{\&c.}\right)\left(\frac{d\psi}{dX}\frac{dX}{dy} + \text{\&c.}\right) + \text{\&c.} = 0.$$

This equation is reduced by the two following identities, which are easily verified,

$$\left(\frac{dX}{dx}\right)^2 + \left(\frac{dX}{dy}\right)^2 + \left(\frac{dX}{dz}\right)^2 = 4X + 4r^2,$$

$$\frac{dX}{dx}\frac{dY}{dx} + \frac{dX}{dy}\frac{dY}{dy} + \frac{dX}{dz}\frac{dY}{dz} = 2(X+Y).$$

The condition may then be written

$$2\left(X\frac{d\phi}{dX} + Y\frac{d\phi}{dY} + \text{\&c.}\right)\left(\frac{d\psi}{dX} + \text{\&c.}\right)$$
$$+ 2\left(X\frac{d\psi}{dX} + \text{\&c.}\right)\left(\frac{d\phi}{dX} + \text{\&c.}\right) + 4\left(r^2\frac{d\phi}{dX}\frac{d\psi}{dX} + \text{\&c.}\right) = 0.$$

The first two groups of terms vanish, because ϕ and ψ, which are satisfied by the coordinates of the point in question, are homogeneous functions of X, Y, &c. The condition therefore is

$$r^2\frac{d\phi}{dX}\frac{d\psi}{dX} + r'^2\frac{d\phi}{dY}\frac{d\psi}{dY} + \text{\&c.} = 0.$$

We may simplify the equations by writing X instead of $X : r$, &c., so that the identity connecting the five spheres becomes

$$X^2 + Y^2 + Z^2 + V^2 + W^2 = 0,$$

and the condition for orthogonal section

$$\frac{d\phi}{dX}\frac{d\psi}{dX} + \frac{d\phi}{dY}\frac{d\psi}{dY} + \text{\&c.} = 0,$$

a condition exactly similar in form to that for ordinary coordinates.

565. We can now immediately, after the analogy of quadrics, form the equation of an orthogonal system of cyclides. For write down the equation

$$\frac{X^2}{\lambda - a} + \frac{Y^2}{\lambda - b} + \frac{Z^2}{\lambda - c} + \frac{V^2}{\lambda - d} + \frac{W^2}{\lambda - e} = 0,$$

in which λ is a variable parameter; and, in the first place, it is easy to see that three cyclides of the system can be drawn through any assumed point: for the equation in λ, though in form of the fourth degree, is in reality only of the third, the coefficient of λ^4 vanishing in virtue of the identical equation. And from the condition just obtained, it follows at once, in the same manner as for confocal quadrics, that any two surfaces of the system cut each other at right angles.*

[The equation $\alpha X + \beta Y + \gamma Z + \delta V = 0$ will represent a point sphere on W, which will also be a focus of the cyclide λ if $\alpha, \beta, \gamma, \delta$ satisfy the two conditions necessary for it to be an enveloping sphere of the cyclide and of W. But these are, as in Art. 560,

$$\frac{\lambda - a}{a - e}\alpha^2 + \frac{\lambda - b}{b - e}\beta^2 + \frac{\lambda - c}{c - e}\gamma^2 + \frac{\lambda - d}{d - e}\delta^2 = 0$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 0.$$

Now it will be found that the values of $\alpha : \beta : \gamma : \delta$ which satisfy these are independent of λ. Hence the above system of]

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* Casey and Darboux seem to have independently made this beautiful extension to three dimensions of Hart's theorem for the corresponding plane curves, *Higher Plane Curves*, Art. 278.

cyclides are confocal, there being a common focal curve on each of the five spheres. It is evident from what has been proved, that confocal cyclides cut each other in their lines of curvature.

[Ex. 1. If λ_1, λ_2, λ_3, be the values of λ corresponding to the members of the family which pass through a point, show that the values of the functions X^2, Y^2, &c., for this point are proportional to $f(a)$, $f(b)$ &c., where

$$f(a) \equiv \{(a - \lambda_1)(a - \lambda_2)(a - \lambda_3)\} \div \psi'(a)$$
$$\psi(\theta) \equiv (\theta - a)(\theta - b)(\theta - c)(\theta - d)(\theta - e).$$

(Use the method indicated at the end of Art. 206.)

Ex. 2. The coordinates of the point are proportional to $\Sigma \frac{\alpha_1}{r_1} \sqrt{f(a)}$, $\alpha_1\beta_1\gamma_1$ being the centre of X, $\alpha_2\beta_2\gamma_2$ that of Y, etc. (Darboux, *op. cit.* note to Art. 515, p. 140).]

566. The mode of generating cyclides as the envelope of a sphere admits of being stated in another useful form. All spheres whose centres lie in a fixed plane, and which meet a given sphere orthogonally, pass through two fixed points, there being two linear relations connecting the coefficients. And it is easy to see what the fixed points are, for since the spheres cut at right angles every sphere through the intersection of the fixed sphere and the plane, they contain the two point-spheres of that system, or the limit points (*Conics*, Art. 111) of the plane and the fixed sphere, these points being real only when the sphere and plane do not intersect in a real curve. In the case, then, where the centre of the movable sphere lies in a fixed surface, it follows, obviously, that the envelope may be described as the locus of the limit points of each tangent plane to the fixed surface and of the fixed sphere. We are thus led to a mode of transformation in which to a tangent plane of one surface answer two points on another; or, if we take the reciprocal of the first surface, it is a (1, 2) transformation, in which to one point on one surface answer two on the other. Dr. Casey has easily proved, p. 598, that the results of substituting the coordinates of one of these limit points in the equations of the spheres of reference are proportional to the perpendiculars let fall from the centres of

these spheres on the tangent plane. Thus, if the surface locus of centres be given by a tangential equation between the perpendiculars from the four centres $\phi(\lambda, \mu, \nu, \rho) = 0$, the derived surface is $\phi(X, Y, Z, W) = 0$; and if the first be the equation of a quadric, the second will be the corresponding cyclide. [If the tangential equation of J, the common orthogonal sphere to X, Y, Z, and W, be $\psi(\lambda, \mu, \nu, \rho) = 0$, it follows by taking ψ as the locus of centres that $J^2 = \psi(X, Y, Z, W)$. Now if the term $\lambda\mu$ is absent from ψ the centres of X and Y are conjugate points with respect to J, and hence X and Y are orthogonal. This proves the converse of the theorem of Art. 563.]

567. From the construction which has been given an analysis has been made by Casey and Darboux of the different forms of cyclides according to the different species of the quadric locus of centres, and the nature of its intersection with the fixed sphere. We only mention the principal cases, remarking in the first place that the spheres whose centres lie along any generator of the quadric all pass through the same circle, namely, that wh'ch has for its anti-points the intersections of the line and the sphere. The circle in question is part of the envelope, which may, therefore, be regarded as the locus of the circles answering to the several right lines of the quadric, there being, of course, two series of circles answering to the two series of right lines.

Now if the quadric be a cone, these circles all lie on the same sphere, that which has its centre at the vertex of the cone and which cuts the given sphere orthogonally, and the cyclide may be regarded as degenerating into the spherical curve which is the envelope of those circles, that curve being the intersection of the sphere by a quadric, which curve has been called a sphero-quartic. Strictly speaking, the cyclide locus of these circles is an annular surface flattened so as to coincide with the spherical area, which is bounded by the sphero-quartic curve. The properties of these sphero-quartics have been investigated in detail by Casey and Darboux.

These curves may be inverted into plane bicircular quartics, and therefore (see note, Art. 515) have four foci, the distances from which to any point of the curve are connected by linear relations.

If the quadric be a paraboloid the cyclide degenerates into a cubic surface passing through the circle at infinity. If the quadric be a sphere the cyclide is the surface of revolution generated by a Cartesian oval round its axis: but Darboux has given the name Cartesian to the more general cyclide generated when the quadric is a surface of revolution.

The cyclide may have one, two, three, or four double points. The nodal cyclides present themselves as the inverse of quadrics, the inverse of the general quadric being a cyclide with one node, that of the general cone one with two, of the general surface of revolution one with three, of the cone of revolution one with four. The last-mentioned, or tetranodal cyclide, is the surface to which the name cyclide was originally given by Dupin, and may therefore be called Dupin's Cyclide. According to its original conception this was the envelope of the spheres, each touching three given spheres; or, more accurately, we have thus four cyclides, for the tangent-spheres in question form four distinct series, those of each series enveloping a cyclide. The spheres of each series are distinguished as having their centres on a given conic; [namely, for a sphere touching S_1, S_2, S_3, all internally or all externaliy, the conic in which the plane $\Sigma(r_2 - r_3)S_1 = 0$ meets the quadrics of type $\sqrt{S_2} - \sqrt{S_3} = \pm (r_2 - r_3)$], and we thus arrive at a better definition; viz. the cyclide is the envelope of a series of spheres each having its centre on a given conic and touching a given sphere.

In the last definition the given sphere is not unique but it forms one of a singly infinite series; in fact, we may, without altering the cyclide, replace the original sphere by any sphere of the series; the new series of spheres have their centres on a conic. It is to be added that instead of the series of spheres having their centres on the first conic, we may obtain the same cyclide as the envelope of a series of spheres having

their centres on the second conic,* and touching a sphere having its centre at any point of the first conic.

The two conics have their planes at right angles, and are such that two opposite vertices of each conic are foci of the other conic; these conics are focal conics of a system of confocal quadric surfaces, one of them is an ellipse and the other a hyperbola.

The relation of the ellipse and hyperbola is such that taking—

(1) Two fixed points on the ellipse, the difference of the distances of these from a variable point on the hyperbola is constant, $= +c$ if the variable point is on one branch, $-c$ if it is on the other branch of the hyperbola (the value of c of course depending on the position of the two fixed points).

(2) Two fixed points on the hyperbola, if on different branches, the sum, but if on the same branch, the difference of their distances from a variable point on the ellipse is constant, the value of this constant, of course, depending on the position of two fixed points.

And using these properties, we see at once how the same surface can be obtained as the envelope of a series of spheres having their centre on either conic, and touching a sphere having its centre at any point of the other conic.

Dupin's Cyclide is also the envelope of a series of spheres having their centres on a conic, and cutting at right angles a given sphere; for instead of the quadric surface in the construction for the general cyclide, we have here a conic.

[567*a*. Nodal cyclides cannot be reduced to the canonical form of Art. 563, for in their case the quadric F and sphere J touch in one or more points, and no common self-conjugate tetrahedron exists. The simplest form of the equation of such cyclides is obtained by choosing such a tetrahedron of reference as will give ϕ and ψ of Art. 566 simple forms.

* [If the contacts are all similar this conic is a hyperbola passing through the centres of S_1, S_2, S_3, and having as its foci the centres of the circles in its plane which touch the spheres.]

By observing that when the term $\lambda\mu$ is absent from ψ, X and Y are orthogonal, and that if the vertex $\lambda = 0$ is on J, X is a point-sphere on J, we can infer the number of nodes and the nature of the inverses of the cyclide from them and derive other properties mentioned above. It will suffice to take one case to illustrate the method, say that in which F and J touch in three distinct points so that their intersection is a circle and two imaginary generators. We may take as simple forms

$$J \equiv \lambda^2 + \mu^2 + \nu^2 + n'\nu\rho \quad F \equiv \lambda^2 + \mu^2 + \nu^2 + n\nu\rho$$

$\nu = 0$ and $\lambda^2 + \mu^2 = 0$ being the points of contact. From these

$$J^2 \equiv X^2 + Y^2 + Z^2 + n'ZW \quad S \equiv X^2 + Y^2 + Z^2 + nZW$$
$$\equiv J^2 + (n - n')\ ZW.$$

The form of S shows that the point-sphere Z, is a node, for being orthogonal to X and to Y it lies on these spheres. Similarly the point-spheres $X + \iota Y$, $X - \iota Y$ are nodes on the surface. We also see that from Z the cyclide inverts into a quadric of revolution, and that the spheres of self-inversion X, Y, J respectively invert into two rectangular planes through the axis of revolution and into the central plane perpendicular thereto. The reader will find it instructive to work out the cases of other geometrical relations between F and J in a similar manner.

The first complete enumeration of the different kinds of cyclides was made by Loria (*Turin Acad.*, 1884), and his eighteen species agree with the number obtained by Segre; of these only ten are real. His researches depend on a system of coordinates in which a sphere plays the part of a line in Plücker's system.

It is clear that if $S_1 \ldots S_5$ are five fixed spheres, $\sum_1^5 x_r S_r = 0$ is an equation which is general enough to denote any sphere; $x_1 \ldots x_5$ are called the coordinates of the sphere, and a single relation between them represents a complex of spheres and two a congruence. Loria shows that a cyclide is the locus of the sphere of a certain congruence consisting entirely of point-spheres.

A slightly different system of spherical coordinates will be found explained in Darboux, *op. cit.*, p. 256.

567*b*. *Quartics with a Cuspidal Conic.*—Let the conic be the intersection of a quadric V with the plane w, then the equation must be $w^3P - V^2 = 0$, when P is a plane. By change of coordinates we may write this without loss of generality,

$$w^3x = (fyz + gzx + hxy + lxw + myw + nzw)^2,$$

and from this form we see that the tangent plane at yzw, an arbitrary point on the conic, touches V and hence envelopes a quadric cone whose vertex is the pole of w with respect to V. The plane x is clearly a trope and any plane through either point of intersection of the trope and conic has a tacnode, as is seen by putting $w = \lambda x + \mu y$, when we get

$$z^2u_1^2 + zu_1u_2 + u_4 = 0,$$

as the form of the equation of the section by any plane through xyw; u_1, u_2, u_4, being binary quantics in x and y. For the tangent plane at the point xyw, u_1 disappears and the equation represents four lines. Hence *a cuspidal conic contains eight lines passing in fours through the two tacnodal points.*]

QUARTICS WITH ISOLATED SINGULARITIES.

568. [Passing now to quartic surfaces without singular lines, the highest singularity they can possess is a triple point. Such surfaces have been studied in detail by Rohn * who has exhaustively classified them according to the different singularities of the tangent cone at the triple point. The memoir contains a number of diagrams illustrating the shapes of the surfaces in the neighbourhood of this point.

These surfaces have for equation the *monoid* †

$$u_3 + \omega u_4 = 0,$$

the twelve lines $u_3 = 0$, $u_4 = 0$ lie on the surface, and the condition that there should be a node C_2 elsewhere is easily found

* *Math. Ann.*, 24.

† Art. 316 (B).

to be that two of these lines should coincide. More generally Rohn shows that the existence of a binode of type B_k involves the coincidence of k of these lines, and that unodes can only occur on a singular edge of the cone u_3. Hence all the types of surfaces for which u_3 is non-singular are obtained by partitioning the number 12; for instance

$$12 = 5 + 3 + 2 + 1 + 1$$

means that there is a surface having, in addition to the triple point, the singularities $B_5 = 1$, $B_3 = 1$, $C_2 = 1$.] Quartic surfaces with isolated double points only may have any number of ordinary conical points up to 16; each such node diminishes the class by 2, so that for the surface with 16 nodes the class is $36 - 2 \times 16, = 4$. Some of the nodes may be replaced by, or may coalesce into, binodes or unodes, but the theory does not appear to have been investigated.

The general cone of contact to a quartic is, by Art. 279, of the twelfth degree, having twenty-four cuspidal and twelve nodal lines, and sixteen is the greatest number of additional nodal lines it can possess without breaking up into cones of lower dimensions. When the surface has sixteen nodes, the cone of contact from each node is of the sixth degree, and has the lines to the other fifteen as nodal lines; from which it follows that this cone breaks up into six planes.

569. It is to be observed that the equation of a quartic surface contains thirty-four constants, that is, the surface may be made to satisfy thirty-four conditions; and that if a given point is to be a node of the surface, this is equivalent to four conditions. It would, therefore, at first sight appear that we could with eight given points as nodes determine a quartic surface containing two constants; but this is not so. We have through the eight points two quadric surfaces $U = 0$, $V = 0$ (every other quadric surface through the eight points being in general of the form $U + \lambda V = 0$) and the form with two constants is in fact $U^2 + \alpha UV + \beta V^2 = 0$, which breaks up into two quadric surfaces, each passing through the eight points. It thus appears that we can find a quartic surface with at most seven given points as nodes.

570. The cases of a surface with 1, 2, or 3 nodes may be at once disposed of; taking, for instance, the first node to be the point (1, 0, 0, 0), the second the point (0, 1, 0, 0), and the third the point (0, 0, 1, 0), we can at once write down an equation $U=0$, with 30, 26, or 22 constants, having the given node or nodes. We might in the same manner take the fourth node to be (0, 0, 0, 1) and write down the equation with 18 constants; but, in the case of four nodes and in reference to those which follow, it becomes interesting to consider how the equation can be built up with quadric functions representing surfaces which pass through the given nodes. In the case of 4 given nodes we have six such surfaces $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$, $U=0$, every other quadric surface through the four points being obtained by a linear combination of these; and we have thence the quartic equation $(P, Q, R, S, T, U)^2=0$ containing apparently 20 constants. The explanation is that the six functions, although linearly independent, are connected by two quadric equations, and the number of constants is thereby reduced to $20-2$, $=18$, which is right. [Taking yz, zx, xy, xw, yw, zw as the six quadrics these are connected by the equations $PS=QT=RU$.]

In the case of 5 given nodes we have through these the five quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$, and we have the quartic surface $(P, Q, R, S, T)^2=0$, containing, as it should do, 14 constants. [One relation connects the quadrics but it is a cubic relation. See Cayley's *Collected Mathematical Papers*, VII. 142.]

571. In the case of 6 given nodes, we have through these the four quadric surfaces $P=0$, $Q=0$, $R=0$, $S=0$, and the quartic surface $(P, Q, R, S)^2=0$ contains only 9 constants; there is in fact through the six points a quartic surface, the Jacobian of the four functions, $J(P, Q, R, S)=0$, not included in the foregoing form, and the general quartic surface with the six given nodes is

$$(P, Q, R, S)^2+\theta J(P, Q, R, S)=0,$$

containing, as it should do, 10 constants.

The foregoing surface $J(P, Q, R, S) = 0$, where $P=0$, $Q=0$, $R=0$, $S=0$ are any quadric surfaces having six common points, is a very remarkable one; it is in fact the locus of the vertices of the quadric cones which pass through the six points. It hereby at once appears that the surface has upon it $15+10, =25$ right lines, namely, the 15 lines joining each pair of the six points, and the 10 lines each the intersection of the plane through three of the points with the plane through the remaining three points. [This is Weddle's surface (Art. 233, note), further discussed in Art. 572*a*.]

In the case of 7 given nodes we have through these three quadric surfaces $P=0$, $Q=0$, $R=0$; but forming herewith the equation $(P, Q, R)^2=0$, this contains only five constants; that it is not the general surface with the 7 given nodes appears also by the consideration that it has, in fact, an eighth node, for each of the intersections of the three quadric surfaces is a node on the surface. We can without difficulty find a quartic surface not included in the form, but having the seven given nodes; for instance, this may be taken to be $\nabla=0$, where ∇ is made up of a cubic surface having four of the points as nodes and passing through the remaining three points, and of the plane through these three points. And the general equation then is

$$(P, Q, R)^2+\theta\nabla=0,$$

containing, as it should do, 6 constants.

572. Passing to the surfaces with 8 nodes, only seven of these can be given points; the eighth may be the remaining common intersection of the quadric surfaces through the seven points, and we thus have a form of surface

$$(P, Q, R)^2=0,$$

with eight nodes, the common intersection of three quadric surfaces; this is the octadic 8-nodal quartic surface. [Octadic surfaces can acquire two additional nodes, see Cayley, VII. 153.]

Among the surfaces of the form in question are included the reciprocals of several interesting surfaces; for example,

order six, parabolic ring; order eight, elliptic ring; order ten, parallel surface of paraboloid, and first central negative pedal of ellipsoid; order twelve, centro-surface of ellipsoid and parallel surface of ellipsoid—the surfaces include also the general torus or surface generated by the revolution of a conic round a fixed axis anywhere situated. [See Cayley, VII. 155.]

There is, however, another kind of 8-nodal surface, called the octo-dianome, for which the eighth node is any point whatever on a certain surface determined by means of the seven given points, and called their dianodal surface.

The last-mentioned surface may be made to have another node, which is any point whatever on a certain curve determined by means of the eight nodes; we have thus the ennea-dianome; and finally this may be made to have a new node, one of a certain system of [thirteen] points determined by means of the nine nodes; this is the deca-dianome. But starting with seven *given* points as nodes, the number of nodes of the quartic surface is at most $=10$.

[The dianodal surface of seven points is obtained by the condition that the 7-nodal quartic should have another node and is $J\{P, Q, R, \nabla\}=0$, a surface of the sixth order. The dianodal surfaces of the seven points 1 to 7 and of the seven 2 to 8 (8 being on the first surface) intersect in the fifteen lines joining 2, 3, 4, 5, 6, 7 and the skew cubic containing these points, and therefore in a residual intersection of the eighteenth order which is the dianodal curve of the eight points. The general forms of octo-dianomes and ennea-dianomes are respectively $(P, Q)^2+\theta\nabla$ and $P^2+\theta\nabla$.]

A kind of 10-nodal surface is the *Symmetroid*, which is represented by means of a symmetrical determinant

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix}=0$$

where the several letters represent linear functions of the coordinates; such a surface has ten nodes [see Art. 572 (*b*)], for

each of which the circumscribed sextic cone breaks up into two cubic cones; and thus the ten nodes form a system of points in space, such that joining any one of them with the remaining nine, the nine lines are the intersections of two cubic cones; these are called an ennead, and the ten points are said to form an enneadic system.

Some of the kinds of surfaces with 11, 12, and 13 nodes, and the surfaces with 14, 15, and 16 nodes were considered by Kummer.* Reverting to the consideration of the circumscribed cone having its vertex at a node, observe that for a surface with 16 nodes, this is a sextic cone with fifteen nodal lines, or it must break up into six planes, say the sextic cone is (1, 1, 1, 1, 1, 1); and the form being unique, this must be the case for the cone belonging to each node of the surface, say the surface is the 16-nodal 16 (1, 1, 1, 1, 1, 1).

Similarly, in the case of 15 nodes, the sextic cone has fourteen nodal lines, or it breaks up into a quadricone and four planes, say it is (2, 1, 1, 1, 1); which form being also unique, the surface is the 15-nodal 15 (2, 1, 1, 1, 1).

In the case of 14 nodes, the cone has thirteen nodal lines, it must be either a nodal cubic cone and three planes, or else two quadricones and two planes; that is (3, 1, 1, 1) or (2, 2, 1, 1). It is found that there is only one kind of surface, having eight nodes of the first sort and six nodes of the second sort; say this is the 14-nodal

$$8\,(3, 1, 1, 1) + 6(2, 2, 1, 1).$$

In the case of 13 nodes, the cones are $(4_3, 1, 1)$, $(3_1, 2, 1)$, $(3, 1, 1, 1)$, or $(2, 2, 2)$, viz. $(4_3, 1, 1)$ is a 3-nodal quartic cone and two planes, and so $(3_1, 2, 1)$ is a nodal cubicone, a quadricone, and a plane. It is found that there are two forms of surface, the 13-(α)-nodal

$$3\,(4_3, 1, 1) + 1\,(3, 1, 1, 1) + 9\,(3_1, 2, 1),$$

and the 13-(β)-nodal 13 (2, 2, 2).

The like principles apply to the cases of twelve, eleven, &c.,

* [And also by Cayley, VII. 279; Rohn, *Math. Ann.*, XXIX., and Jessop, *Quart. J.*, 31.]

nodes, [but for their discussion the reader is referred to Rohn's memoir mentioned in the footnote to this article].

[572*a*. *Some Properties of Weddle's Surface.*—Let C be the skew cubic through the nodes $(N_1 \ldots N_6)$ and let $(P \,.\, C)$ denote the cone joining a point P to C.

1. *Any plane through two nodes cuts the surface in a cubic on which the nodes are corresponding points.* For the nodal cone at N_1 contains N_1N_r, and therefore is $(N_1 \,.\, C)$. The plane meets C in one point P other than the nodes, and the lines joining P to the nodes are generators of each nodal cone and therefore touch the plane cubic of section at the nodes.

2. Let PN_1, PN_2 meet the quartic again in P_1, P_2. Then from the properties of a plane cubic N_2P_1 and N_1P_2 meet on the quartic, say in P_{12}. We derive thus six points P_r and fifteen points P_{rs}. Baker shows that N_qP_{rs} and N_rP_{qs} meet on the quartic in a point P_{qrs}, that P_{123} is the same point as P_{456}, and thus we have *a closed system of thirty-two points on the quartic lying in pairs on ninety-six lines through the nodes.*

3. Weddle's Surface may be defined as the locus of points whose polar planes with respect to all quadrics through the six nodes are concurrent, and in this sense is made up of pairs of corresponding points. Now if P and Q are such a pair the line PQ must be a common tangent line at P to all quadrics through $N_1 \ldots N_6$ and P, or in other words, the eighth common point of these quadrics coincides with P. This leads to the definition: *Weddle's Surface is the locus of a point which is itself the eighth intersection of all quadrics through it and six fixed points.*

4. *The line PQ meets C twice and is divided harmonically by it*, for three independent quadrics can be described through the nodes and the two points where the chord of C through P meets C.

5. If the co-ordinates of a point of C are taken to be $1 : \lambda : \lambda^2 : \lambda^3$, and if λ_r is the parameter of N_r, and we put

$$f(x) \equiv \{(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)(x-\lambda_4)(x-\lambda_5)(x-\lambda_6)\}^{-\frac{1}{2}}$$

then it is easy to see from the preceding theorem that *the coordinates of a point on Weddle's Surface may be expressed in terms of two parameters as follows*

$$x : y : z : w :: f(\theta) \pm f(\phi) : \theta f(\theta) \pm \phi f(\phi) : \theta^2 f(\theta) \pm \phi^2 f(\phi) : \theta^3 f(\theta) \pm \phi^3 f(\phi).$$

For other properties see the papers cited in Art. 545 from which the foregoing are taken.

572*b*. *Symmetroids.*—The existence of ten nodes is connected with the existence of ten plane-pairs of the *syzygy* of quadrics

$$\alpha U_1 + \beta U_2 + \gamma U_3 + \delta U_4 = 0. \quad (U_r = a_r x^2 + \&c.).$$

For from the four equations obtained by differentiating we can either eliminate $\alpha, \beta, \gamma, \delta$ and obtain the Jacobian (J); or eliminate x, y, z, w and obtain, considering $\alpha, \beta, \gamma, \delta$ as coordinates, a symmetroid (S) as written in Art. 572, where

$$a = \alpha a_1 + \beta a_2 + \gamma a_3 + \delta a_4,$$

and so on. By means of the four equations the coordinates of a point on J are given as rational functions of a point on S, and *vice versa*, so *there is a one-one correspondence of points on S and J.*

But the values of $\alpha, \beta, \gamma, \delta$, which make $\alpha U_1 + \&c.$ a plane-pair make every first minor of S zero, so that to the whole edge, which lies in J, corresponds a node on S; thus the ten nodes correspond to the ten edges.

If $a = 0$ the point $h = g = l$ is an eleventh node on S. This will be the case if the four quadrics all pass through the point yzw, and therefore J has a node there.

Making $a = b = c = d = 0$ we see that the symmetroid with fourteen nodes, may be written $\sqrt{fl} + \sqrt{gm} + \sqrt{hn} = 0$, and corresponds to the Jacobian of four quadrics through four points.

If we make these quadrics have one, or two, more common points, we obtain symmetroids with 15 or 16 nodes. In these cases the planes f, g, &c., must be connected by one or two equations of the form

$$\eta\zeta f + \zeta\xi g + \xi\eta h + \xi\omega l + \eta\omega m + \xi\omega n \equiv 0$$

obtained by adding the four equations, which express that a point ξ, η, ζ, ω, lies on the quadrics u_1, u_2, u_3, and u_4, these equations being first multiplied by α, β, γ, and δ respectively. We thus obtain the important result proved directly by Jessop * that *the* 14, 15, *and* 16 *nodal quartics are all included in the form*

$$\sqrt{xX} + \sqrt{yY} + \sqrt{zZ} = 0,$$

subject in the case of 15 *and* 16 *nodal quartics to one or two conditions of the form*

$$Ax + By + Cz + DX + EY + FZ \equiv 0$$
$$\text{where } AD = BE = CF.]$$

[573. The 16-nodal or Kummer's Quartic was discussed in Art. 455a in connexion with quadratic complexes, but it will be instructive to consider this surface from an independent point of view.† We have seen in Art. 572 that the enveloping cone from each node breaks up into six planes intersecting on the fifteen lines joining the node in question to the others. It is easy to see that these planes touch the tangent cone at the node along the six lines of closest contact. Each of these planes contains six nodes, namely, the original one and one in each of the other five planes, and as six planes pass through each node there are sixteen such planes altogether. Further, these planes are tropes; for the section of the quadric by any one of them has the six nodes lying on it as double points and must accordingly be a repeated conic through the nodes, for the only other plane section of a quartic having six double points is made up of four right lines, and this hypothesis is inadmissible as no three nodes are collinear. Thus, then, *the sixteen nodes lie by sixes on conics in sixteen tropes, and the tropes touch in sixes the quadric tangent cones at the nodes.* This arrangement of points and planes is known as a 16_6 *configuration.*

* *Quarterly Journal*, vol. 31.

† The subject-matter of this and the next three Articles is largely taken from Hudson, *op. cit.* Art. 545.

Denoting one node by 0, the tropes through it by the numbers 1 to 6, and the other nodes by the binary symbols 12, 13, &c., we can now see how these latter are distributed on the ten remaining tropes. For two triads of tropes such as 1, 2, 3 and 4, 5, 6 cut any plane in two triangles circumscribed to the section by that plane of the nodal cone at 0, and therefore the six vertices of these triangles lie on a conic. In other words, the lines joining 0 to 12, 23, 31, 45, 56, 64 lie on a quadric cone. There are ten such cones, and only ten in general, corresponding to the ten partitions of the six tropes into a pair of triads. But the six nodes lying on any other trope connect to 0 by the edges of a quadric cone, which cone must be identical with one of the ten just mentioned; and hence the six nodes 12, 23, 31, 45, 56, 64 lie on a trope.

We may call this the trope $\frac{123}{456}$ and can now at once specify the tropes through any node; for instance, those through 12 are $1, 2, \frac{123}{456}, \frac{124}{356}, \frac{125}{346}, \frac{126}{345}$.

573*a*. As the general quartic surface contains thirty-four constants in its equation, and as the existence of a node implies one relation between these, we would expect that the equation of Kummer's Quartic should contain eighteen independent constants, and that thus six arbitrary points can be selected as nodes.

The notation of the preceding article enables us to verify this and to see that *six arbitrary points, no four of which are coplanar, determine a* 16_6 *configuration in twelve different ways.* For let us call one definite point 0 and a second 12, which is clearly permissible since no two nodes have any special relation to each other. If to the other four points are attached the symbols 23, 34, 45, 51, nothing is implied except that no four of the six points are coplanar; and this allocation of symbols can be made in *twenty-four* ways.

This being done, the ten planes

$$1, 2, 3, 4, 5, \frac{123}{456}, \frac{234}{156}, \frac{345}{126}, \frac{145}{236}, \frac{125}{346}$$

are determined, for three points in each are known, and the ten other points will be found to be determined each as the intersection of some three of these ten planes. The reduction to *twelve* distinct ways depends on the fact that to call the four points 51, 45, 34, 23 instead of 23, 34, 45, 51 will not lead to a different configuration; for the same ten additional points will be determined though their symbols will be changed by the interchange of 1 with 2 and 3 with 5, e.g. 25 becomes 13 and so on.

573*b*. *Analytical treatment of the* 16_6 *configuration.* Let the symbol a denote the operation of interchanging y with z and x with w, and let b and c have similar meanings. Further, let A, B, C denote the operations of changing the signs of y and z, of z and x, and of x and y respectively, while d denotes that no interchange, and D that no change of signs, is made. From the equation

$$\alpha x + \beta y + \gamma z + \delta w = 0,$$

which we call dD, fifteen others are derived by combining some one of the operations a, b, c, d, with A, B, C or D, and we will call these equations by the symbol of the operation whereby they are derived. These sixteen equations may be taken as representing at the same time the planes or points of a 16_6 configuration; for instance the point bC represented by $\alpha w + \beta z - \gamma y - \delta x = 0$ is the point $(-\delta, -\gamma, \beta, \alpha)$ which lies on the plane $\alpha x + \beta y + \gamma z + \delta w = 0$, that is the plane dD; thus the six points in the plane dD are aB, aC, bC, bA, cA, cB. The six points that lie in the plane aB are dD, dA, cA, cC, bC, bD, a result that can be obtained immediately by operating on the previous six points with aB and remembering that $a^2 = d$, $ab = c$, and so on.

The relations between the points and planes are given by the following diagram; the six planes (or points) passing through (or lying on) a given point (or plane) are those whose symbols appear in the same row or column as the symbol of the given point (or plane):—

$$\begin{array}{cccc} dD & aB & bC & cA \\ aC & dA & cD & bB \\ bA & cC & dB & aD \\ cB & bD & aA & dC \end{array}$$

Eighteen constants are involved in the foregoing equations, namely, three to fix each plane of reference, three to fix the ratios of the coordinates, and the three ratios $\alpha:\beta:\gamma:\delta$. They are accordingly sufficiently general to represent any 16_6 configuration.

573*c*. The equation of Kummer's Quartic which has the points and planes of last article for nodes and tropes can now be written down. For it must be unchanged by any of the operations a, b, c, A, B, C, and is therefore of the form

$$x^4+y^4+z^4+w^4+L\ (y^2z^2+x^2w^2)+M\ (z^2x^2+y^2w^2) + N\ (x^2y^2+z^2w^2)+2Kxyzw=0.$$

The condition that $(\alpha, \beta, \gamma, \delta)$ is a node gives the equations

$$L=\frac{\beta^4+\gamma^4-\alpha^4-\delta^4}{\alpha^2}$$

with two similar ones, and

$$K=\frac{\alpha\beta\gamma\delta(\delta^2+\alpha^2-\beta^2-\gamma^2)(\delta^2+\beta^2-\gamma^2-\alpha^2)(\delta^2+\gamma^2-\alpha^2-\beta^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)}{(\alpha^2\delta^2-\beta^2\gamma^2)(\beta^2\delta^2-\gamma^2\alpha^2)(\gamma^2\delta^2-\alpha^2\beta^2)}.$$

If α, β, γ, δ be eliminated from these we obtain the condition already found in Art. 455*e*, viz.—

$$4-L^2-M^2-N^2+LMN+D^2=0.$$

If δ be put equal to zero the equation of the surface takes the particular form

$$x^4+y^4+z^4+w^4-2p\ (y^2z^2+x^2w^2)-2q\ (z^2x^2+y^2w^2) -2r\ (x^2y^2+z^2w^2)=0$$

where p, q, and r are the cosines of the angles of a plane triangle whose sides are α^2, β^2, and γ^2. In this case the sixteen nodes lie in fours in the coordinate planes, and the section of the surface by any one of these planes is a pair of conics intersecting in the nodes and having as the sides of their common self-conjugate triangle the traces of the other coordinate planes.

Such a surface is called a *Tetrahedroid*, and includes the Wave-surface as a particular case, as can be at once seen by writing in the equation just given when multiplied by $a^2b^2c^2$

$$-x^2,\ -y^2,\ -z^2,\ 1,\ -(b^2+c^2),\ -(c^2+a^2),\ -(a^2+b^2)$$

in place of

$$bcx^2,\ cay^2,\ abz^2,\ w^2,\ 2bcp,\ 2caq,\ 2abr$$

respectively, when we obtain the ordinary equation of the Wave-surface.

Ex. 1. The coordinates of a point of Kummer's Surface are expressible in terms of two parameters as follows :—

$$x:y:z:w::1:\tfrac{1}{2}(u+v):uv:(u-v)^{-2}\left(\sqrt{u_1u_2u_3v_4v_5v_6}+\sqrt{v_1v_2v_3u_4u_5u_6}\right)^2$$

where $\qquad u_r \equiv u - k_r$ and $v_r \equiv v - k_r$.

Let 0, 0, 0, 1 be a node, $xz - y^2 = 0$, the nodal cone at it and the six planes of the type

$$\pi_r \equiv k_r^2x - 2k_ry + z = 0$$

be the six tropes through it. Let $w = 0$ be the trope $\frac{123}{456}$.

Writing the equation of the surface in the form

$$w^2(xz - y^2) + wu_3 + \phi^2 = 0 \quad . \quad . \quad . \quad . \quad \text{(i)}$$

we require

$$(xz - y^2)\phi^2 - u_3^2 \equiv \lambda\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6 \quad . \quad . \quad . \quad \text{(ii)}$$

Any tangent to the section of $xz - y^2$ by w has as its equation $\theta^2x - 2\theta y + z = 0$, and if u and v be the values of θ corresponding to the tangents from any point we have

$$x:y:z::1:\tfrac{1}{2}(u+v):uv.$$

The equation $(u_1u_2u_3v_4v_5v_6 - u_4u_5u_6v_1v_2v_3)(u-v)^{-1} = 0$ is satisfied for the six nodes in w, and as it is symmetrical in u and v and u^2v^2 is the highest term, it represents the conic through them, $\phi = 0$. Substituting in (ii) we get

$$x^{-6}u_3^2 = (u_1u_2u_3v_4v_5v_6 - v_1v_2v_3u_4u_5u_6)^2 + \lambda u_1u_2u_3u_4u_5u_6v_1v_2v_3v_4v_5v_6$$

whence $\lambda = 4$, for u_3 must be a symmetrical function of u and v. Solve finally for $w : x$ from (i).

Ex. 2. Six of the sixteen planes of Art. 573*b* are the planes corresponding to the point $(\alpha, \beta, \gamma, \delta)$ with reference to the six mutually *apolar* * linear complexes $p \pm s = 0$, $q \pm t = 0$, $r \pm s = 0$. They are the six called in that Article aB, aC, bC, bA, cA, cB.

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* Two linear complexes $Ap + Bq + \&c. = 0$, $A'p + B'q + \&c. = 0$ are said to be *apolar* when their coefficients satisfy the relation

$$AD' + A'D + BE' + B'E + CF' + C'F = 0.$$

In this case the conjugate line with respect to one complex of any ray of the second is also a ray of the second.

Ex. 3. The ten other planes are the polar planes of $(\alpha, \beta, \gamma, \delta)$ with respect to ten quadrics thus defined: take any three of the linear complexes of Ex. 2, and their common rays will be one system of generators of one of the quadrics while the common rays of the three remaining complexes will be the other system of generators of the same quadric. The equations of the quadrics are $x^2 + y^2 + z^2 + w^2 = 0$; $y^2 + z^2 - x^2 - w^2 = 0$ and two similar; $yz \pm xw = 0$ and four similar; the results can be easily verified by means of these equations.

Ex. 4. The preceding examples, together with the fact that six mutually apolar linear complexes can always be reduced to the forms of Ex. 2 by choosing as edges of the tetrahedron of reference the directing lines (Art. 455) of the three congruences obtained by combining the complexes into three pairs, lead to the following theorem:—*Six mutually apolar linear complexes and one arbitrary point determine uniquely a* 16_6 *configuration.*

574. Very few investigations concerning non-singular quartic surfaces have been published. The following is a slight sketch of a method by which Rohn * has proved that *the maximum number of ovals which a non-singular quartic surface may possess is ten.* By an *oval* portion of a surface he means one without singularities and closed. That quartics may actually possess as many as ten ovals appears from the equation

$$XYZW - k^2(x^2 + y^2 + z^2 - r^2) - c^2 = 0$$

in which X, Y, Z, and W are, when equated to zero, the Cartesian equations, in the standard form of Art. 27, of the faces of a regular tetrahedron whose centre is the origin and the distance of whose corners from the origin exceeds r. If c were zero this equation would represent a 12-nodal quartic having six identical portions lying outside the tetrahedron in the acute angle between each pair of faces and four identical portions lying inside the tetrahedron and near the corners, these ten portions being joined at the nodes but otherwise distinct. Now suppose a small value to be given to c, and we obtain ten oval portions lying within the former portions.

The proof that ten is the maximum number depends on somewhat intricate considerations of the tangent cone to a 10-nodal quartic from one of the nodes and cannot be given here. But it is easy to see that this number cannot exceed

* *Lupziger Berichte*, LXIII. (1911).

twelve. For if possible let $S=0$ be a quartic with thirteen ovals, and suppose that the sign of the function S is positive for points inside the ovals. Let $V=0$ be a quadric which does not meet all the ovals in real curves and consider the pencil of quartics $S-kV^2=0$, as k grows from zero.

The members of this pencil are quartics with thirteen ovals lying within those of S, up to such a value of k as makes one of the ovals shrivel up into a point, A, which will be a conjugate point on that particular quartic. Consider the tangent cone from A to this quartic. It will be of the sixth order and be such that any plane cuts it in twelve ovals.

But this is impossible; for if a plane sextic had twelve ovals we could choose one point on eleven of them and three on the twelfth and through these fourteen points describe a plane quartic which would meet the eleven ovals in two points each and the twelfth in four and therefore the sextic in twenty-six. Hence the quartic surface can only have at most twelve ovals.

The process of replacing ovals by conjugate points is called a *Schrumpfungsprozess* (shrivelling) by Rohn, and he shows that any number of ovals up to ten can be thus replaced one by one, the remaining ovals lying within the original ones. In fact the only difference is that the quadric V is made to pass through the conjugate points already obtained, so that $S-kV^2$ preserves these singularities and is made to acquire one more conjugate point at each step.]

CHAPTER XVII.

GENERAL THEORY OF SURFACES.

SECTION I.—SYSTEMS OF SURFACES.

575. We shall in this chapter proceed, in continuation of Art. 287, with the general theory of surfaces, and shall first give for surfaces in general a few theorems proved for quadrics (Art. 233, &c.).

The locus of the points whose polar planes with regard to four surfaces U, V, W, T (whose degrees are m, n, p, q) meet in a point, is a surface of the degree $m+n+p+q-4$; the Jacobian of the system, which is also the locus of all double points of the system $\lambda U+\mu V+\nu W+\rho T$. For its equation is evidently got by equating to nothing the determinant whose constituents are the four differential coefficients of each of the four surfaces. If a surface of the form $\lambda U+\mu V+\nu W$ touch T, the point of contact is evidently a point on the Jacobian, and must lie somewhere on the curve of the degree $q\,(m+n+p+q-4)$ where the Jacobian meets T. In like manner, $pq\,(m+n+p+q-4)$ surfaces of the form $\lambda U+\mu V$ can be drawn so as to touch the curve of intersection of T, W; for the point of contact must be some one of the points where the curve TW meets the Jacobian.

It follows hence, that the tact-invariant of a system of three surfaces U, V, W (that is to say, the condition that two of the mnp points of intersection may coincide), contains the coefficients of the first in the degree $np\,(2m+n+p-4)$; and in like manner for the other two surfaces. For, if in this condition we substitute for each coefficient a of U, $a+\lambda a'$, where a' is the corresponding coefficient of another surface U' of the same

degree as U, it is evident that the degree of the result in λ is the same as the number of surfaces of the form $U + \lambda U'$, which can be drawn to touch the curve of intersection of V, W.*

I had arrived at the same result otherwise thus: (see *Quarterly Journal*, vol. I. p. 339). Two of the points of intersection coincide if the curve of intersection UV touch the curve UW. At the point of contact then the tangent planes to the three surfaces have a line in common; and these planes therefore have a point in common with any arbitrary plane $\alpha x + \beta y + \gamma z + \delta w$. Thus the point of contact annuls the determinant, which has for one row, α, β, γ, δ; and for the other three, the four differentials of each of the three surfaces. The condition that this determinant may vanish for a point common to the three surfaces is got by eliminating between the determinant and U, V, W. The result will contain α, β, γ, δ in the degree mnp; and the coefficients of U in the degree $np\,(m+n+p-3)+mnp$. But this result of elimination contains as a factor the condition that the plane $\alpha x + \beta y + \gamma z + \delta w$ may pass through one of the points of intersection of U, V, W. And this latter condition contains α, β, γ, δ in the degree mnp, and the coefficients of U in the degree np. Dividing out this factor, the quotient, as already seen, contains the coefficients of U in the degree

$$np\,(2m+n+p-4).$$

576. The locus of points whose polar planes with regard to three surfaces have a right line common is, as may be inferred from the last article, the Jacobian curve denoted by the system of determinants

$$\left\| \begin{matrix} U_1, & U_2, & U_3, & U_4 \\ V_1, & V_2, & V_3, & V_4 \\ W_1, & W_2, & W_3, & W_4 \end{matrix} \right\| = 0.$$

But this curve (see *Higher Algebra*, Art. 271) is of the degree

$$(m'^2 + n'^2 + p'^2 + m'n' + n'p' + p'm'),$$

where m' is the degree of U_1, &c., that is to say, $m' = m - 1$, &c.

* Moutard, *Terquem's Nouvelles Annales*, xix. p. 58.

If a surface of the form $\lambda U + \mu V$ touch W, the point of contact is evidently a point on the Jacobian curve, and therefore the number of such surfaces which can be drawn to touch W is equal to the number of points in which this curve meets W, that is to say, is p times the degree of that curve. Reasoning then, as in the last article, we see that the tact-invariant of two surfaces U, V, that is to say, the condition that they should touch, contains the coefficients of U in the degree

$$n(n'^2 + 2m'n' + 3m'^2),$$

or

$$n(n^2 + 2mn + 3m^2 - 4n - 8m + 6).$$

This number may be otherwise expressed as follows: if the degree and class of V be M and N, and the degree of the tangent cone from any point be R, then the degree in which the coefficients of U enter into the tact-invariant is

$$N + 2R(m-1) + 3M(m-1)^2.$$

We add, in the form of examples, a few theorems to which it does not seem worth while to devote a separate article.

Ex. 1. Two surfaces, U, V of degrees m, n intersect; the number of tangents to their curve of intersection, which are also inflexional tangents of the first surface, is $mn(3m + 2n - 8)$.

The inflexional tangents at any point on a surface are generating lines of the polar quadric of that point; any plane therefore through either tangent touches that polar quadric. If then we form the condition that the tangent plane to V may touch the polar quadric of U, which condition involves the second differentials of U in the third degree, and the first differentials of V in the second degree, we have the equation of a surface of degree $(3m + 2n - 8)$ which meets the curve of intersection in the points, the tangents at which are inflexional tangents on U.

Ex. 2. In the same case to find the degree of the surface generated by the inflexional tangents to U at the several points of the curve UV.

This is got by eliminating $x'y'z'w'$ between the equations

$$U' = 0,\ V' = 0,\ \Delta U' = 0,\ \Delta^2 U' = 0,$$

which are in $x'y'z'w'$ of degrees respectively m, n, $m-1$, $m-2$, and in $xyzw$ of degrees 0, 0, 1, 2. The result is therefore of degree $mn(3m-4)$.

Ex. 3. To find the degree of the developable which touches a surface along its intersection with its Hessian. The tangent planes at two consecutive points on the parabolic curve intersect in an inflexional tangent (Art. 269); and, by the last example, since $n = 4(m-2)$, the degree of the surface generated by these inflexional tangents is $4m(m-2)(3m-4)$. But since at every point of the parabolic curve the two inflexional tangents coincide, and therefore the surfaces generated by each of these tangents coincide, the number just found must be divided by two, and the degree required is $2m(m-2)(3m-4)$.

Ex. 4. To find the characteristics, as in Art. 330, of the developable circumscribed along any plane section to a surface whose degree is m. The section of the developable by the given plane is the section of the given surface, together with the tangents at its $3m(m-2)$ points of inflexion. Hence we easily find degree $= 6m(m-2)$, class $= m(m-1)$, $r=m(3m-5)$, $\alpha=0$, $\beta=2m(5m-11)$, &c.

Ex. 5. To find the characteristics of the developable which touches a surface of degree m along its intersection with a surface of degree n. We find

class $= mn(m-1)$, $\alpha = 0$, $r = mn\ (3m + n - 6)$, whence the other singularities are found as in Art. 330.

Ex. 6. To find the characteristics of the developable touching two given surfaces, neither of which has multiple lines. We find

class $= mn\ (m-1)^2\ (n-1)^2$, $\alpha = 0$, $r = mn\ (m-1)\ (n-1)\ (m+n-2)$.

Ex. 7. To find the characteristics of the curve of intersection of two developables.

The surfaces are of degrees r and r', and since each has a nodal and cuspidal curve of degrees respectively x and m, x' and m', therefore the curve of intersection has $rx' + r'x$ and $rm' + r'm$ actual nodal and cuspidal points. The cone therefore which stands on the curve, and whose vertex is any point, has nodal and cuspidal edges in addition to those considered at Art. 343; and the formulæ there given must then be modified. As there the degree is rr'; but the degree of the reciprocal of this cone is

$$rr'\ (r + r' - 2) - r\ (2x' + 3m') - r'\ (2x + 3m),$$

or, by the formulæ of Art. 327, rank $= rn' + r'n$. In like manner

$$\text{class} = \alpha r' + \alpha' r + 3rr'.$$

Ex. 8. To find the characteristics of the developable generated by a line meeting two given curves. This is the reciprocal of the last example. We have therefore class $= rr'$, rank $= rm' + r'm$, degree $= \beta r' + \beta' r + 3rr'$.

Ex. 9. To find the characteristics of the curve of intersection of a surface and a developable. The letters M, N, R relate to the surface as in the present article; m, n, r to the developable. We find degree $= Mr$, rank $= rR + nM$, class $= \alpha M + 3rR$.

Ex. 10. To find the characteristics of a developable touching a surface and also a given curve. We find degree $= \beta N + 3rR$, rank $= rR + mN$, class $= Nr$.

577. The theory of *systems of curves* given in *Higher Plane Curves*, p. 372, obviously admits of extension to surfaces. Let it be supposed that we are given one less than the number of conditions necessary to determine a surface of degree n; the surfaces satisfying these conditions form a system whose characteristics are μ, ν, ρ; where μ is the number of surfaces of the system which pass through any point, ν is the number which touch any plane, and ρ the number which touch any line. It is obvious that the sections of the

system of surfaces by any plane form a system of curves whose characteristics are μ, ρ; and the tangent cones drawn from any point form a system whose characteristics are ρ, ν. Several of the following theorems given by De Jonquières (*Comptes Rendus*, LVIII., p. 567), answer to theorems already proved for curves.

(1) *The locus of the poles of a fixed plane with regard to surfaces of the system is a curve of double curvature of degree* ν. The locus is a curve, since the plane itself can only be met by the locus in a finite number of points ν. Taking the plane at infinity, we find, as a particular case of the above, the locus of the centre of a quadric satisfying eight conditions. Thus, when eight points are given, the locus is a curve of the third degree; when eight planes, it is a right line.

(2) *The envelope of the polar planes of a fixed point, with regard to all the surfaces of the system, is a developable of class* μ.

(3) *The locus of the poles with regard to surfaces of the system, of all the planes which pass through a fixed right line, is a surface of degree* ρ. There are evidently ρ and only ρ points of the locus, which lie on the assumed line. The theorem may otherwise be stated thus: understanding by the polar curve of a line with respect to a surface, the curve common to the first polars of all the points of the line; then, *the polar curves of a fixed line with regard to all the surfaces of the system lie on a surface of degree* ρ.

(4) Reciprocally, *The polar planes of all the points of a line, with respect to surfaces of the system, envelope a surface of class* ρ.

(5) *The locus of the points of contact of lines drawn from a fixed point to surfaces of the system is a surface of degree* $\mu+\rho$, *having the fixed point as a multiple point of order* μ. This is proved as for curves. The problem may otherwise be stated: "To find the locus of a point such that the tangent plane at that point to one of the surfaces of the system which passes through it shall pass through a fixed point." Hence we may infer the locus of points, where a given plane is cut

orthogonally by surfaces of the system. It is the curve in which the plane is cut by the locus surface $\mu+\rho$, answering to the point at infinity on a perpendicular to the given plane.

(6) *The locus of points of contact, with surfaces of the system, of planes passing through a fixed line, is a curve of degree $\nu+\rho$ meeting the fixed line in ρ points.* This also may be stated as the locus of a point, the tangent plane at which to one of the surfaces of the system passing through it contains a given line.

(7) *The locus of a point such that its polar plane with regard to a given surface of degree m, and the tangent plane at that point to one of the surfaces of the system passing through it, intersect in a line which meets a fixed right line, is a surface of degree $m\mu+\rho$.* The locus evidently meets the fixed line in the ρ points where it touches the surfaces of the system, and in the m points where it meets the fixed surface, these last being multiple points on the locus of order μ.

(8) *If in the preceding case the line of intersection is to lie in a given plane, the locus will be a curve of degree $m(m-1)\mu+m\rho+\nu$.* The ν points where the fixed plane is touched by surfaces of the system are points on the locus; and also the points where the section of the fixed surface by the fixed plane is touched by the sections of the surfaces of the system. But the number of these last points is $m(m-1)\mu+m\rho$.

The locus just considered meets the fixed surface in $m\{m(m-1)\mu+m\rho+\nu\}$ points. But it is plain that these must either be the $m(m-1)\mu+m\rho$ points just mentioned, or else points where surfaces of the system touch the fixed surface. Subtracting, then, from the total number the number just written, we find that—

(9) *The number of surfaces of the system which touch a fixed surface is $m(m-1)^2\mu+m(m-1)\rho+m\nu$*; or, more generally, if n be the class of the surface, and r the degree of the tangent cone from any point, the number is $n\mu+r\rho+m\nu$.

We can hence determine the number of surfaces of the form $\lambda U+V$ which can touch a given surface. For if U and

V are of degree m, these surfaces form a system for which $\mu = 1$, $\nu = 3(m-1)^2$, $\rho = 2(m-1)$. If, then, n be the degree of the touched surface, the value is

$$n\,(n-1)^2 + 2n\,(n-1)\,(m-1) + 3n\,(m-1)^2,$$

the same value as that given, Art. 576. This conclusion may otherwise be arrived at by the following process.

578. *If there be in a plane two systems of points having a (n, m) correspondence, that is, such that to any point of the first system correspond m of the second, and to any point of the second correspond n of the first: and, moreover, if any right line contains r pairs of corresponding points, then the number of points of either system which coincide with points corresponding to them is $m+n+r$.* Let us suppose that the coordinates of two corresponding points xy, $x'y'$, are connected by a relation of the degrees μ, μ' in xy, $x'y'$ respectively; and by another relation of the degrees ν, ν'; then if $x'y'$ be given, there are evidently $\mu\nu$ values of xy, hence $n = \mu\nu$. In like manner $m = \mu'\nu'$. If we eliminate x, y between the two equations, and an arbitrary equation $ax + by + c = 0$, we obtain a result of the degree $\mu\nu' + \mu'\nu$ in $x'y'$: showing that if one point describe a right line, the other will describe a curve of the degree $\mu\nu' + \mu'\nu$, which will, of course, intersect the right line in the same number of points, hence $r = \mu\nu' + \mu'\nu$. But if we suppose x' and y' respectively equal to x and y, we have $(\mu + \mu')\,(\nu + \nu')$ values of x and y; a number obviously equal to $m + n + r$. [This proof assumes that the sets of m and n points are complete intersections, but a general proof can be given by means of the principle of correspondence.]

579. Let us now proceed to investigate the nature of the locus of points, whose polar planes with respect to surfaces of the system coincide with their polars with respect to a fixed surface; and let us examine how many points of this locus can lie in an assumed plane. Let there be two points A and a in the plane, such that the polar plane of A with respect to the fixed surface coincides with the polar plane of a with

respect to surfaces of the system. Now, first, if A be given, its polar plane with regard to the fixed surface is given; and the poles of that plane with respect to surfaces of the system lie, by theorem (1), on a curve of degree ν. This curve will meet the assumed plane in the points a which correspond to A, whose number therefore is ν. On the other hand, if a be given, its polar planes with respect to surfaces of the system envelope, by theorem (2), a developable whose class is μ; but the polar planes of the points of the given plane with regard to the fixed surface envelope a surface whose class is $(m-1)^2$;* this surface and the developable have common $\mu(m-1)^2$ tangent planes, which will be the number of points A corresponding to a. Lastly, let A describe a right line, then its polar planes with respect to the fixed surface envelope a developable of the class $m-1$; but with respect to the surfaces of the system, by theorem (4), envelope a surface of the class ρ. There may, therefore, be $\rho\,(m-1)$ planes whose poles on either hypothesis lie on the assumed line. Hence, last article, the number of points A which coincide with points a is $\mu\,(m-1)^2+\rho\,(m-1)+\nu$. The locus, then, of points whose polar planes with respect to the system, and with respect to a fixed surface, coincide, will be a curve of the degree just written, and it will meet the fixed surface in the points where it can be touched by surfaces of the system.

580. We add a few more theorems given by De Jonquières.

(10) *The locus of a point such that the line joining it to a fixed point, and the tangent plane at it to one of the surfaces of the system which pass through it, meet the plane of a fixed curve of degree m in a point and line which are pole and polar with respect to that curve, is a curve of degree* $\mu m\,(m-1)+\rho m+\nu$.

This is proved as theorem (8). Let the fixed curve be the imaginary circle at infinity, and the theorem becomes *the*

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* It was mentioned (Art. 524) that if the equation of a plane contain two parameters in the degree n, its envelope will be of the class n^2.

locus of the feet of the normals drawn from a fixed point to the surfaces of the system is a curve of degree $2\mu + 2\rho + \nu$.

(11) If there be a system of plane curves, whose characteristics are μ, ν, the locus of a point such that its polar with regard to a fixed curve of degree m, and the tangent at it to one of the curves of the system which pass through it, cut a given finite line harmonically, is a curve whose degree is $m\mu + \nu$. Consider in how many points the given line meets the locus, and evidently its ν points of contact with curves of the system are points on the locus. But, reasoning as in other cases, we find that there will be m points on the line, which together with their polars with respect to the fixed curve divide the given line harmonically. And since these are points on the locus for each of the μ curves which pass through them, the degree of the locus is $m\mu + \nu$. Taking for the finite line the line joining the two imaginary circular points at infinity, it follows that there are $m(m\mu + \nu)$ curves of the system which cut a given curve orthogonally. De Jonquières finds that in like manner *the locus of a point such that its polar plane with regard to a fixed surface, and the tangent plane at that point to one of the surfaces of the system, meet the plane of a fixed conic in two lines conjugate with respect to the conic, is a surface of degree* $m\mu + \rho$. And consequently that a surface of this degree meets the fixed surface in points where it is cut orthogonally by surfaces of the system.

(12) If from each of two fixed points Q, Q' tangents be drawn to a system of plane curves of class n, the locus of the intersections of the tangents of one system with those of the other is a curve of degree $\nu(2n - 1)$. For consider any curve touching the line QQ', then one point of the locus will be the point of contact, and $n - 1$ of the others will coincide with each of the points Q, Q'. And since there may be ν such curves, each of the points Q, Q', is a multiple point of order $\nu(n - 1)$, and the line QQ' meets the locus in $\nu(2n - 1)$ points. Let the points QQ' be the two circular points at infinity, and it follows that the locus of foci of curves of the system is a curve of degree $\nu(2n - 1)$. If we investigate, in like manner,

the locus of the intersection of cones drawn to a system of surfaces from two fixed points QQ', it is evident, from what has been said, that any plane through QQ' meets the locus in a curve whose degree is $\rho(2n-1)$; but the line QQ' is a multiple line of order ρ, being common to both cones in every case where the line QQ' touches a surface of the system. *The degree of the locus therefore is* $2n\rho$: and accordingly, 4ρ is the degree of the locus of foci of sections of a system of quadrics by planes parallel to a fixed plane.

Chasles has given the theorem that if there be a system of conics whose characteristics are μ, ν, then $2\nu-\mu$ conics of the system reduce to a pair of lines, and $2\mu-\nu$ to a pair of points. It immediately follows hence, as Cremona has remarked, that if there be a system of quadrics, whose characteristics are μ, ν, ρ, of which σ reduce to cones and σ' to plane conics, then considering the section of the system by any plane, we have $\nu=2\rho-\mu$, $\sigma'=2\mu-\rho$, and, reciprocally, $\sigma=2\nu-\rho$. These theorems, however, are obviously subject to modifications if it can ever happen that a surface of the system can reduce to a pair of planes or a pair of points. Thus in the simple case of the system through six points and touching two planes, the ten pairs of planes through the six points are to be regarded as surfaces of the system, since a pair of planes is a quadric which touches *every* plane. For the same reason the problem to describe a quadric through six points to touch three planes does not, as might be thought, admit of 27 but only of 17 solutions, the ten pairs of planes counting among the apparent solutions.

I have attempted to enumerate the number of quadrics which satisfy nine conditions, *Quarterly Journal*, VIII. 1 (1866). The same problem has been more completely dealt with by Chasles and Zeuthen (see *Comptes Rendus*, Feb. 1866, p. 405).

SECTION II.—TRANSFORMATIONS OF SURFACES.

581. The theory of the *transformation* of curves and of the *correspondence* of points on curves (explained *Higher*

Plane Curves, Chap. VIII.) is evidently capable of extension to space of three dimensions, but only a very slight sketch can here be given of what has been done on this subject.

It will be recollected that a unicursal curve is a curve, the points of which have a (1, 1) correspondence with those of a straight line; or, analytically, we can express the coordinates x, y, z of a point of it as proportional to homogeneous functions, of the same degree m, of two parameters λ, μ. Similarly, a unicursal surface is a surface, the points of which have a (1, 1) correspondence with those of a plane; or, analytically, we can express the coordinates x, y, z, w of any of its points as proportional to homogeneous functions, of the same degree m, of three parameters λ, μ, ν. When the points of a surface have thus a (1, 1) correspondence with those of a plane, it is evident that every curve on the surface corresponds in the same manner to a curve in the plane, which latter curve may, therefore, be taken as a representation (*Abbildung*) of the former curve.

582. It is geometrically evident that quadrics and cubics are unicursal surfaces. If we project the points of a quadric on a plane by means of lines passing through a fixed point O on the surface, we obtain at once a (1, 1) correspondence between the points of the quadric and of the plane. In the case of the cubic, taking any two of the right lines on the surface, any point on the surface may be projected on a plane by means of a line meeting the two assumed lines, and we have in this case also a (1, 1) correspondence between the points of the surface and of the plane. From the construction in the case of the quadric can easily be derived analytical expressions giving x, y, z, w as quadratic functions of three parameters. And such expressions can be obtained in several other ways: for instance, coordinate systems have been formed by Plücker and Chasles (see Art. 393), determining each point on the surface by means of the two generators which pass through it. And, indeed, the method by which the generators are expressed by means of parameters (Art. 108)

at once suggests a similar expression for the coordinates of a point (see Art. 421, Ex. 2) on the surface. Thus, on the quadric $xw = yz$, the systems of generators are $\lambda x = \mu y$, $\mu w = \lambda z$; $\lambda x = \nu z$, $\nu w = \lambda y$, whence the coordinates of any point on the quadric may be taken $\mu\nu$, $\lambda\nu$, $\lambda\mu$, λ^2. The construction we have indicated in the case of a cubic may also be used to furnish expressions for the coordinates in terms of parameters; but other methods effect the same object more simply. For instance, Clebsch has used the theorem that any cubic may be generated as the locus of the intersection of three corresponding planes, each of which passes through a fixed point. If A, B, C; A', B', C'; A'', B'', C'' represent planes, we evidently obtain the equation of a cubic by eliminating λ, μ, ν between the equations

$$\lambda A + \mu B + \nu C = 0, \quad \lambda A' + \mu B' + \nu C' = 0, \quad \lambda A'' + \mu B'' + \nu C'' = 0;$$

and if we take λ, μ, ν as parameters, we can evidently, by solving these three equations for x, y, z, w, which they implicitly contain, obtain expressions for the coordinates of any point on the cubic, as cubic functions of the three parameters.

583. It will be more simple, however, if we proceed by a converse process. Let us suppose that we are given a system of equations $x : y : z : w = P : Q : R : S$, where P, Q, R, S are homogeneous functions, of degree m, of three parameters λ, μ, ν. This system of equations evidently represents a surface, the equation of which can be found by eliminating λ, μ, ν from the equations, when there results a single equation in x, y, z, w. If λ, μ, ν be taken as the coordinates of a point in a plane, the given system of equations establishes a (1, 1) correspondence between the points of the surface and of the plane. $P = 0$, &c., denote curves of degree m in that plane. Let us first examine the degree of the surface represented by the system of equations, or the number of points in which it is met by an arbitrary line $ax + by + cz + dw$, $a'x + b'y + c'z + d'w$. To these points evidently correspond in the plane the intersections of the two curves

$$aP + bQ + cR + dS = 0, \quad a'P + b'Q + c'R + d'S = 0,$$

whence it follows that the degree of the surface is in general m^2. If, however, the curves P, Q, R, S have a common points,* the two curves have besides these only $m^2 - a$ other points of intersection, and accordingly this is the degree of the surface. Then to any plane section of the surface will correspond in the plane a curve $aP + bQ + cR + dS$ passing through the a points: these two curves will have the same deficiency, and we are thus in each case enabled to determine whether a plane section of the surface contains multiple points, that is to say, whether the surface contains multiple lines. To the section of the surface, by a surface of degree k, $ax^k + \&c. = 0$ corresponds in the plane a curve $aP^k + \&c. = 0$ of degree mk, and on this each of the a points is a multiple point of order k. Again, the given system of equations determines a point on the surface corresponding to each point of the plane, except in the case of any of the a points. For each of these, the expressions for x, y, z, w vanish, and their mutual ratios become indeterminate: to one of these points then corresponds on the surface not a point, but a locus, which will ordinarily be a right line on the surface. To a curve of degree p on the plane will correspond on the surface a curve the degree of which (that is to say, the number of points in which it is met by an arbitrary plane) is the same as the number of points in which the given plane curve is met by a curve $aP + bQ + cR + dS$. This number will be, in general, mp, but it will be reduced by one for each passage of the given curve through one of the a points.

584. In conformity, then, with the theory thus explained, let P, Q, R, S be quadratic functions of λ, μ, ν; then $P = 0$, &c. represent conics; and in order that the corresponding surface should be a quadric, it is necessary and sufficient that the conics P, Q, R, S should have two common points A, B. Then to any point in the plane ordinarily corresponds a point on the surface, except that to the points A, B correspond right lines

* For simplicity, we only notice the case where the common points are ordinary points, but of course some of them may be multiple points.

on the surface. To a plane section of the quadric corresponds in general a conic passing through AB; but this conic may in some cases break up into the line AB, together with another line; and in fact the previous theory shows that to every right line in the plane thus corresponds in general a conic on the quadric. If, however, the line in the plane pass through either of the points A, B, the corresponding locus on the quadric is only of the first degree, and we are thus by this method led to see the existence of two systems of lines on the surface, the lines of one system all meeting a fixed line A, those of the other a fixed line B.

585. If the conics P, Q, R, S have but one common point A, the surface is a cubic; but as each plane section of the cubic corresponds to a conic, and is therefore unicursal, it must have a double point, and the cubic surface has a double line. And since to every line through the point A corresponds a line on the surface, we see that the cubic is a ruled surface. In like manner, if P, Q, R, S have no common point, the surface is a quartic; but every plane section being unicursal, the quartic has a double curve of the third degree; this is either Steiner's surface already referred to or a ruled surface.

586. Again, let P, Q, R, S be cubic functions of λ, μ, ν; in order that the surface represented should be a cubic, the curves P, Q, R, S must have six common points. Then the deficiency of the curve $aP + \&c.$ being unity, this is also the deficiency of a plane section of the cubic; that is to say, the surface has no double line. To the six points will correspond six non-intersecting lines on the surface; these will be one set of the lines of a Schläfli's double-six.

To a line in the plane corresponds on the surface a skew cubic curve, but if the line pass through one of the six points, the corresponding curve will be a conic, and if the line join two of the six points, the corresponding curve will be a right line. We thus see that there are on the surface, in addition to the six lines with which we started, fifteen others,

each meeting two of the six lines. Again, to a conic in the plane corresponds in general a sextic curve on the surface, but this will reduce to a line if the conic pass through five of the six points. We have thus six other lines on the surface, each meeting five of the original six; and thus the entire number is made up of $27 = 6 + 15 + 6$.

Suppose, however, P, Q, R, S to be still cubic functions, but that the curves represented by them have only five common points, then, by the previous theory, the surface represented is a quartic, but the deficiency of a plane section being unity, the quartic must have a double conic. There will be on the quartic right lines, viz. five corresponding to the five common points, one corresponding to the conic through these points, and ten to the lines joining each pair of the points; or sixteen in all (see Art. 559). This is the method in which Clebsch arrived at this theory (*Crelle*, LXIX. p. 142).

587. The deficiency or genus of a plane curve of degree n with δ double points and κ cusps is $\frac{1}{2}(n-1)(n-2) - \delta - \kappa$; it is equal to the number of arbitrary constants contained (homogeneously) in the equation of a curve of degree $n-3$, which passes through the $\delta + \kappa$ double points and cusps; and it was found by Clebsch that there is a like expression for the "deficiency" of a surface of degree n having a double and a cuspidal curve; it is equal to the number of arbitrary constants contained (homogeneously) in the equation of a surface of degree $n-4$, which passes through the double and cuspidal curves of the given surface.* Prof. Cayley thence deduced the expression

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)(b+c) \\ + \tfrac{1}{2}(q+r) + 2t + \tfrac{7}{2}\beta + \tfrac{5}{2}\gamma + i - \tfrac{1}{8}\theta,$$

* More generally, if the surface has an i-ple curve and also j-ple points, then it is found by Dr. Noether (*An. di Mat.* (2), v. p. 163) that the deficiency is equal to the number of constants, as above, in the equation of a surface of degree $n-4$, which passes $(i-1)$ times through the i-ple curve (has this for an $(i-1)$-ple line), and $(j-2)$ times through each j-ple point (has this for a $(j-2)$-ple point).

where b, q are the degree and class of the double curve, c, r those of the cuspidal curve, t the number of triple points on the double curve, β, γ, i the number of intersections of the two curves (β of those which are stationary points on the double curve, γ stationary points on the cuspidal curve, i not stationary on either curve), and θ the number of singularities of a certain other kind. In the case where there is only a double curve without triple points the formula is

$$D = \tfrac{1}{6}(n-1)(n-2)(n-3) - (n-3)b + \tfrac{1}{2}q.$$

Thus in the several cases,

Quadric surface	$n=2,\ b=0,\ q=0$
General cubic surface	$n=3,\ b=0,\ q=0$
Quartic with double right line	$n=4,\ b=1,\ q=0$
,, ,, ,, conic	$n=4,\ b=2,\ q=2$
Quintic with a pair of non-intersecting double right lines	$n=5,\ b=2,\ q=0$
Quintic with a double skew cubic	$n=5,\ b=3,\ q=4$

and in all these cases we find $D=0$ or the surface is unicursal.

CREMONA TRANSFORMATIONS.

[587*a*. As in two dimensions (see *Higher Plane Curves*, ch. VIII.) the most powerful method of studying algebraic surfaces of high degree is to establish a (1, 1) correspondence between their points and those of a simpler surface. Such a correspondence is called *birational*, since when the coordinates of a point on either surface are given, those of the corresponding point are determined algebraically and uniquely, and therefore rationally; the surfaces are said to be *transformed into* or *represented upon* one another. The representation of a unicursal surface upon a plane is an important example; it follows from the definitions that any surface which can be birationally transformed into a unicursal surface is itself unicursal.

Let the two surfaces $f(x, y, z, w)$, $F(X, Y, Z, W)$, supposed to exist in unconnected spaces, be in birational correspondence. Then if $a(x, y, z, w)$ and $A(X, Y, Z, W)$ are

corresponding points, the coordinates of A are determined, when those of a are given, by equations of the form

$$X : Y : Z : W = \phi_1 : \phi_2 : \phi_3 : \phi_4, \qquad . \quad . \quad (1)$$

where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are homogeneous functions of x, y, z, w of degree n say, which is the degree of the transformation (1). Then if we eliminate x, y, z, w from the equation $f=0$ by means of the equations (1) we obtain the equation $F=0$ of the second surface.

If A is an arbitrary point in the second space, not lying on F, then (1) represents three surfaces of degree n in the first space, which have in general n^3 common points; so that, regarded as a correspondence between the whole of the two spaces, the transformation is not birational, but of order n^3. But if A lies on F, then one and in general only one of the n^3 points lies on f, and this one is the point a, uniquely determined. That is to say, the equations (1) are not rationally reversible by themselves, but if we join to them the equation $f=0$, we can express the coordinates of a rationally in terms of those of A in the form

$$x : y : z : w = \Phi_1 : \Phi_2 : \Phi_3 : \Phi_4, \qquad . \quad . \quad (2)$$

where Φ_1, Φ_2, Φ_3, Φ_4 are homogeneous functions of X, Y, Z, W of degree N say. The transformation (2) is the *inverse* of (1).

587*b*. But if, for all positions of A, the three surfaces given by (1) have fixed points or curves in common, the number of variable points in the first space corresponding to A is less than n^3, and if the common elements are such as to absorb all but one of the variable points of intersection, we have a (1, 1) correspondence between the whole of the two spaces; this is called a *Cremona* transformation. Then the equations (1) can be reversed without the aid of the equation of the surface, and are exactly equivalent to (2), and to every surface f in one space there corresponds a surface F in the other.

The family of surfaces (ϕ) whose general member is

$$\phi \equiv a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + a_4\phi_4$$

corresponds to all the planes in the second space; this family is linear and triply infinite, and the necessary and sufficient

condition for a Cremona transformation is that any three members shall have one and only one point of intersection which varies with the parameters a_1, a_2, a_3, a_4. A family of surfaces which satisfies these conditions is called *homaloidal*; every homaloidal family gives rise to a Cremona transformation, in which it corresponds to the family of planes in the other space, and therefore every *homaloid*, or member of a homaloidal family, is unicursal. The same remarks apply to the family (Φ) in the second space.

Any point common to all the surfaces ϕ, whether an isolated point or lying on a common curve, is an exception to the (1, 1) correspondence. The equations to determine the coordinates of the corresponding point cease to have meaning in the form (1), and in the form (2) they cease to be independent, and give a locus, either a curve or a surface, instead of a single point.

Now let there be a curve c of degree m common to all the ϕ's; then any two ϕ's have a variable curve of intersection γ of degree $n^2 - m$, which corresponds to a straight line in the second space. To the $n^2 - m$ intersections of γ with any plane correspond the intersections of a straight line with any Φ, whose number is N. Therefore the degree of the inverse transformation is

$$N = n^2 - m.$$

Thus if $n = 1$, N must be 1; if $n = 2$, then N can be 2, 3, or 4, etc.

There are $\frac{1}{6}(n+1)(n+2)(n+3)$ terms in the general homogeneous function of degree n in four variables, and therefore as many linearly independent surfaces of degree n, and any three of these meet in n^3 points. In order to select a homaloidal family, we must impose on the general surface conditions whose *postulation*, or the number of independent relations between the coefficients to which they give rise, is $\frac{1}{6}(n+1)(n+2)(n+3) - 4$, and whose *equivalence*, or the number of variable points of intersection which they absorb, is $n^3 - 1$. Guccia (*Rend. C.M. Palermo*, I. p. 339) has shown that the second of these statements includes the first; it follows

that none of the conditions can be such as to increase the postulation without increasing the equivalence (e.g. the condition of dividing a given segment harmonically), for if it were so, by releasing this condition we should have a family which satisfied the condition of equivalence, but whose postulation was too low. Thus the conditions all impose upon the family common elements, whose aggregate is called the *fundamental system*. These fundamental elements may be either points or curves, either simple or of given multiplicity; and the surfaces may be required to pass through the element, or to have contact of a given order with a given surface at the point or along the curve, or to have several sheets passing through the element, with different conditions of contact for the different sheets. Further complications arise from the intersections of different fundamental curves, or the passage of these curves through points of higher singularity. It is impossible at present to give any general formulæ for either postulation or equivalence, but some partial results have been obtained (Cayley, *Coll. Math. Papers*, VII. p. 189; Noether, *An. di Mat.* (2) V. p. 163; Hudson, *Proc. Lond. Math. Soc.* (2), XI. p. 398).

587*c*. Before going further with the general theory, let us discuss a simple example. There is a quadratic transformation whose inverse is also quadratic, $n = N = 2$, and the degree of the fundamental curve is $n^2 - N = 2$. This cannot be two skew straight lines, for their equivalence is found to be 8, which is too great; it is therefore a conic c. Then the postulation is 5 and the equivalence 6, and each is too low by 1; therefore there is also one simple fundamental point a.

Let the equations of c be $w = 0$, $q(x, y, z) = 0$, and let the coordinates of a be (0, 0, 0, 1). Then the quadric homaloidal family is

$$\phi \equiv (a_1x + a_2y + a_3z)w + a_4q,$$

and the equations of transformation are

$$X : Y : Z : W = xw : yw : zw : q,$$

and of the inverse,

$$x : y : z : w = XW : YW : ZW : Q,$$

where Q is the same function of X, Y, Z that q is of x, y, z, so that the second homaloidal family is of the same nature as the first, with a fundamental conic C, $W=0$, $Q=0$, and a fundamental point A $(0, 0, 0, 1)$.

To a general surface $f(x, y, z, w)$ of degree n there corresponds the surface $f(XW, YW, ZW, Q)$, or say $F(X, Y, Z, W)$, of degree $2n$. Every term of F has a factor of the form W^aQ^{n-a}, and therefore the surface has C as an n-fold conic, and similarly it has A as an n-fold point. But if f passes through a, then every term of f contains either x, y, or z, and after substitution W is a factor, and the remaining factor F is of degree $2n-1$. Now WF has C, A as n-fold elements, and as W passes once through C and not through A, therefore F passes $n-1$ times through C and n times through A; the passage of f through a lowers by 1 the degree of the transformed surface and also the multiplicity of C on it. The whole plane W corresponds to the point a, and F corresponds to the rest of f. F meets W in the conic C counted $n-1$ times and in a residual straight line; the part of F near the residual corresponds to the part of f near a, or we may say that in the limit, the residual corresponds to the tangent plane of f at a. If f passes through c, the factor Q is to be dropped from the equation of the corresponding surface. This factor represents the cone vertex A standing on C; each of its generators corresponds to a single point of c. For example, let $q \equiv yz+zx+xy$, then the point b $(0, 1, 0, 0)$ lies on c. If f passes through b, each term of f contains x, z, or w as a factor, and each term of F contains XW, ZW, or $YZ+ZX+XY$, and therefore the surface F contains the whole of the straight line $X=Z=0$, which is a generator of the cone Q and corresponds to b. Similarly, if f passes s times through a and t times through c, then the factor W^sQ^t is dropped, the transformed surface is of degree $2n-s-2t$, and on it A is an $(n-2t)$-fold point and C is an $(n-s-t)$-fold curve; the residual intersection of F with W is a curve of degree s corresponding to the tangent cone of f at a.

Inversely, every point of the plane w corresponds to A.

Since the quadrics (ϕ) meet w in the conic c, a ϕ can contain no other point of w unless it degenerates and contains every point; hence every point of w presents the same condition ($a_4 = 0$) to the family (ϕ) and corresponds to the same point A in the second space, which is a fundamental point, since there is not one definite point corresponding to it. Since every straight line meets w in one point, the variable conic Γ of intersection of any two Φ's passes once through A, which is a simple fundamental point. In the same way, a generator l of the cone q meets ϕ in a and one point of c and in no other point, unless it lies altogether on ϕ; every point of l presents the same condition to ϕ and corresponds to the same point B in the second space. Now ϕ meets the cone q in c and two generators of the cone; i.e. every ϕ contains two positions of l, and every plane in the second space contains two positions of B, whose locus is therefore a fundamental conic, in fact C.

These surfaces w, q are the *principal* elements, corresponding to the fundamental elements in the other space. The Jacobian of the homaloidal family is given by a numerical multiple of w^2q, and is entirely composed of principal surfaces.

By means of this transformation we can study a variety of surfaces. For example, consider a quartic surface f having a double conic c. Take c and any point a of the surface as the fundamental system of a quadro-quadric transformation; then f is transformed into a cubic surface F. All curves on f are transformed into curves on F, except that if f contains a straight line l through a, since f meets the plane of c in c counted twice and in no other point, therefore l also meets c and is transformed into a point; but all the curves upon a cubic surface are known, and hence we can discover all those that exist upon f, as well as other properties of the surface.

The quadro-quadric transformation can be specialised in two ways: (1) the conic degenerates, (2) the point lies on the conic.

(1) If c is a pair of lines, we may assume $q = yz$; then the cone Q also degenerates into the pair of planes YZ, one cor-

responding to each of the lines of c. If any surface f contains only one of the lines, the degree of the transformed surface is lowered by 1. The inverse transformation is specialised in just the same way.

(2) The fundamental point a cannot lie in the plane of the conic c without lying on c, for otherwise the whole homaloidal family would degenerate and consist not of quadrics but of planes. But if a moves up to a point of c, in the limit ϕ touches a fixed plane at a; this condition replaces that of passing through the isolated point. If c does not degenerate, let its equations now be $x=0$, $yw+z^2=0$; let the coordinates of a be (0, 0, 0, 1), and let the fixed tangent plane at a be $y=0$. Then ϕ has the form

$$(a_1x+a_2y+a_3z)x+a_4(yw+z^2)$$

and the equations of transformation are

$$X:Y:Z:W=x^2:xy:xz:yw+z^2$$

$$x:y:z:w=XY:Y^2:YZ:XW-Z^2.$$

The cone q degenerates into the plane of c and the fixed tangent plane at a. If a surface f passes through a, the degree of the corresponding surface F is lowered by 1; if f also touches y at a, the factor Y^2 is dropped, and the degree of F is lowered by 2.

587*d*. The only other homaloidal families of quadrics are found to be those with a fundamental straight line and three points, and those with three simple points and a point of contact. The inverse transformations are of degrees 3 and 4 respectively.

Of higher transformations, we may briefly refer to the cubo-cubic transformation given by three equations linear in each set of coordinates, which has occupied a large space in the literature of the subject. The fundamental system in each space consists of a sextic curve of genus 3; to the points of each of these curves correspond the trisecants of the other, which generate a scroll of degree 8, the Jacobian of the family. The sextic may degenerate in a great variety of ways, which are usually different in the two spaces. If it

consists of the six edges of a tetrahedron, the cubic homaloids all have four double points, and the equations take the elegant form

$$Xx = Yy = Zz = Ww.$$

The simplest transformation of degree n is monoidal (Art. 316) when the homaloids have a common multiple point of order $n-1$. In particular, let $w\phi + \psi$ be any monoid, where ϕ, ψ are functions of x, y, z only; this forms part of a homaloidal family, of which the other independent members are $x\theta$, $y\theta$, $z\theta$. For the equations

$$X : Y : Z : W = x\theta : y\theta : z\theta : w\phi + \psi$$

are rationally reversible in the form

$$x : y : z : w = X\Phi : Y\Phi : Z\Phi : W\Theta - \Psi.$$

Since we can always find a monoid through any given curve, this gives a Cremona transformation that changes any given curve into a plane curve; the usual method of projection is a birational transformation of the curve alone, and not of the whole space.

587*e*. Now let us return to the general theory, and consider what is the principal system corresponding to the fundamental system. To a point a lying on all the surfaces (ϕ) there corresponds either a surface or a curve. If it is a surface J', which meets every straight line, then a lies on the variable intersection γ of every pair of ϕ's, i.e. a is either an isolated fundamental point, or a multiple point lying on a fundamental curve of lower multiplicity, and the degree of J' is equal to the number of branches of γ through a. But if a is an ordinary point of a fundamental curve c, then γ does not in general pass through a, and corresponding to a we have not a surface but a curve K. The degree of K, or the number of its intersections with any plane, is equal to the multiplicity i of c on (ϕ). As a moves on c, this curve K generally moves and describes a surface J'', whose degree is equal to the number of intersections of c, γ. If m is the degree of c, or the number of intersections of c with any plane, then any Φ meets J'' in m curves such as K, and the remaining part of the inter-

section of Φ, J'' consists of fundamental curves only. But if γ does not meet c, then to every point a of c there corresponds the same curve C, which does not describe a surface, but is fixed. Then every point of C corresponds to every point of c, and C is also a fundamental curve, of degree and multiplicity equal respectively to the multiplicity and degree of c.

It can be shown that J' is a unicursal surface, and that its different points are in (1, 1) correspondence with the different directions issuing from the point to which J' corresponds. The surfaces such as J', J'', taken with proper multiplicities, make up the Jacobian of (Φ), which can be defined as the locus of double points of Φ's; this can be proved by showing that a double point, if it lies upon J' or J'', presents less than four conditions to (Φ), and is therefore possible, and conversely that a possible position for a double point must correspond to a fundamental point in the other space.

All the points of J' (or of K) present the same condition to (Φ), and any Φ which contains one such point contains them all. If a Φ degenerates, one component corresponds to the plane in the other space, and the other component does not correspond to a surface, but to a curve or point, and therefore is part of the Jacobian.

To a general surface $f(x, y, z, w)$ of degree t there corresponds the surface $f(\Phi_1, \Phi_2, \Phi_3, \Phi_4)$, or say $F(X, Y, Z, W)$, of degree tN, and containing the second fundamental system repeated t times; but if f passes through a fundamental element, the corresponding factor J is dropped from the transformed equation, and the degree of F is lowered. To a general curve of degree t there corresponds a curve of degree tN, which again is lowered if the first curve meets a fundamental curve or passes through a fundamental point.

587*f*. Segre (*An. di Mat.* (2), xxv. p. 2) has used the quadric transformation to analyse the higher singularities of surfaces into their constituents. We have seen that if a is an s-fold point on f, a quadric transformation with a as fundamental point transforms f into a surface having, corresponding to a,

a plane curve of degree s. If the tangent cone at a has multiple edges, the curve has multiple points, which may be multiple points of the surface. The order of these points is in general less than s, but it may be as high as s, if the tangent cone at a consists of s planes meeting in a line; in every case, a finite number of quadric transformations reduces the singularity to a set of multiple points of lower order.]

SECTION III.—CONTACT OF LINES WITH SURFACES.

588. We now return to the class of problems proposed in Art. 272, viz. to find the degrees of the curves traced on a surface by the points of contact of lines which satisfy three conditions, and of the scrolls generated by such lines. The cases we shall consider are:—

(A) flecnodal lines, which meet the surface in four consecutive points;

(B) lines which are inflexional tangents at one point and ordinary tangents at another;

(C) triple tangent lines, which are ordinary tangents at three points.

Now to commence with problem A; if a line meet a surface U in four consecutive points, the tangent plane meets U in a curve having at the point a flecnode, or node having there an inflexion on one branch; the tangent to this branch is the flecnodal line. We must at the point of contact not only have $U'=0$, but also $\Delta U'=0$, $\Delta^2 U'=0$, $\Delta^3 U'=0$. The tangent line must then be common to the surfaces denoted by the last three equations.

But since the six points of intersection of these surfaces are all coincident with $x'y'z'w'$, the problem is a case of that treated in Art. 473. Since then, by that article, the condition $\Pi=0$, that the three surfaces should have a common line, is of degree

$$\lambda'\lambda''\mu+\lambda''\lambda\mu'+\lambda\lambda'\mu''-\lambda\lambda'\lambda'';$$

substituting

$$\lambda=1,\ \lambda'=2,\ \lambda''=3;\ \mu=n-1,\ \mu'=n-2,\ \mu''=n-3;$$

we find that Π is of degree $11n-24$. *The points of contact then of lines which meet the surface in four consecutive points lie on the intersection of the surface with a derived surface S of degree* $11n-24$.*

The intersection of this surface S with the given surface U is a curve of degree $a_4 = n\,(11n-24)$, "the flecnodal curve" of U.

589. We proceed to give Clebsch's calculation, determining the equation of this surface S which meets the given surface at the points of contact of lines which meet it in four consecutive points. It was proved, in the last article, that in order to obtain this equation it is necessary to eliminate between the equations of an arbitrary plane and of the surfaces $\Delta U'$, $\Delta^2 U'$, $\Delta^3 U'$. This elimination is performed by solving for the coordinates of the two points of intersection of the arbitrary plane, the tangent plane $\Delta U'$, and the polar quadric $\Delta^2 U'$; substituting these coordinates successively in $\Delta^3 U'$, and multiplying the results together. Let the four coordinates of the point of contact be x_1, x_2, x_3, x_4; the running coordinates y_1, y_2, y_3, y_4; the differential coefficients u_1, u_2, u_3, u_4; the second and third differential coefficients being denoted in like manner by suffixes, as u_{12}, u_{123}. Through each of the lines of intersection of $\Delta U'$, $\Delta^2 U'$, we can draw a plane, so that by suitably determining t_1, t_2, t_3, t_4, we can, in an infinity of ways, form an equation identically satisfied

$$\Delta^2 U' + (t_1y_1 + t_2y_2 + t_3y_3 + t_4y_4)\,\Delta U'$$
$$= (p_1y_1 + p_2y_2 + p_3y_3 + p_4y_4)\,(q_1y_1 + q_2y_2 + q_3y_3 + q_4y_4) \quad . \; . \; . \quad \text{(I)}.$$

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* I gave this theorem in 1849 (*Cambridge and Dublin Journal*, IV. p. 260). I obtained the equation in an inconvenient form (*Quarterly Journal*, I. p. 336); and in one more convenient ((*Philosophical Transactions*, CL. 1860, p. 229) which I shall presently give. But I substitute for my own investigation the very beautiful piece of analysis by which Prof. Clebsch performed the elimination indicated in the text, *Crelle*, LVIII. p. 93. Prof. Cayley has observed that exactly in the same manner as the equation of the Hessian is the transformation of the equation $rt - s^2$ which is satisfied for every point of a developable, so the equation $S = 0$ is the transformation of the equation (Art. 437) which is satisfied for every point on a ruled surface.

We shall suppose this transformation effected; but it is not necessary to determine the actual values of t_1, &c., for it will be found that these quantities disappear from the result. Let the arbitrary plane be $c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4$, then it is evident that the coordinates of the intersections of the arbitrary plane, the tangent plane $u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4$, and $\Delta^2 U'$, are the four determinants of the two systems

$$\left\|\begin{matrix} c_1, & c_2, & c_3, & c_4 \\ u_1, & u_2, & u_3, & u_4 \\ p_1, & p_2, & p_3, & p_4 \end{matrix}\right\|, \qquad \left\|\begin{matrix} c_1, & c_2, & c_3, & c_4 \\ u_1, & u_2, & u_3, & u_4 \\ q_1, & q_2, & q_3, & q_4 \end{matrix}\right\|.$$

These coordinates have now to be substituted in $\Delta^3 U'$, which we write in the symbolical form $(a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4)^3$; where a_1 means $\frac{d}{dx_1}$, &c., so that, after expansion, we may, for any term $a_1a_2a_3y_1y_2y_3$, substitute $u_{123}y_1y_2y_3$, &c. It is evident then that the result of substituting the coordinates of the first point in $\Delta^3 U'$ may be written as the cube of the symbolical determinant $\Sigma a_1c_2u_3p_4$, where, after cubing, we are to substitute third differential coefficients for the powers of the a's as has been just explained. In like manner, we write the result of substituting the coordinates of the second point $(\Sigma b_1c_2u_3q_4)^3$, where b_1 is a symbol used in the same manner as a_1. The eliminant required may therefore be written

$$(\Sigma a_1c_2u_3p_4)^3\,(\Sigma b_1c_2u_3q_4)^3 = 0.*$$

The above result may be written in the more symmetrical form

$$(\Sigma a_1c_2u_3p_4)^3\,(\Sigma b_1c_2u_3q_4)^3 + (\Sigma b_1c_2u_3p_4)^3\,(\Sigma a_1c_2u_3q_4)^3 = 0.$$

For, since the quantities a, b are after expansion replaced by differentials, it is immaterial whether the symbol used originally were a or b; and the left-hand side of this equation when

* The reason why we use a different symbol for $\frac{d}{dx_1}$, &c., in the second determinant is because if we employed the same symbol, the expanded result would evidently contain sixth powers of a, that is to say, sixth differential coefficients. We avoid this by the employment of different symbols, as in Prof. Cayley's "Hyperdeterminant Calculus" (*Lessons on Higher Algebra*, Lesson XIV.), with which the method here used is substantially identical.

expanded is merely the double of the last expression. We have now to perform the expansion, and to get rid of p and q by means of equation (I). We shall commence by thus banishing p and q.

590. Let us write

$$F=(\Sigma a_1c_2u_3p_4)\,(\Sigma b_1c_2u_3q_4),\ G=(\Sigma b_1c_2u_3p_4)\,(\Sigma a_1c_2u_3q_4).$$

The eliminant is $F^3+G^3=0$, or $(F+G)^3-3FG\,(F+G)=0$. We shall separately examine $F+G$, and FG, in order to get rid of p and q. If the determinants in F were so far expanded as to separate the p and q which they contain we should have

$$F=(m_1p_1+m_2p_2+m_3p_3+m_4p_4)\,(n_1q_1+n_2q_2+n_3q_3+n_4q_4),$$
$$G=(n_1p_1+n_2p_2+n_3p_3+n_4p_4)\,(m_1q_1+m_2q_2+m_3q_3+m_4q_4),$$

where, for example, m_4 is the determinant $\Sigma a_1c_2u_3$, and n_4 is $\Sigma b_1c_2u_3$. If then i, j be any two suffixes, the coefficient of m_in_j in $F+G$ is $(p_iq_j+p_jq_i)$. And we may write

$$F+G=\Sigma\Sigma m_in_j\,(p_iq_j+p_jq_i),$$

where both i and j are to be given every value from 1 to 4. But, by comparing coefficients in equation (I), we have

$$p_iq_j+p_jq_i=2u_{ij}+(t_iu_j+t_ju_i),$$

whence $\qquad F+G=2\Sigma\Sigma m_in_ju_{ij}+\Sigma\Sigma m_in_j(t_iu_j+t_ju_i)$.

Now it is plain that if for every term of the form $p_iq_j+p_jq_i$ we substitute $t_iu_j+t_ju_i$, the result is the same as if in F and G we everywhere altered p and q into t and u. But, if in the determinants $\Sigma a_1c_2u_3q_4$, $\Sigma b_1c_2u_3q_4$ we alter q into u, the determinants would vanish as having two columns the same. The latter set of terms therefore in $F+G$ disappears, and we have $\frac{1}{2}(F+G)=\Sigma\Sigma m_in_ju_{ij}$.

Now, if we remember what is meant by m_i, n_j, this double sum may be written in the form of a determinant

$$-\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & a_1, & c_1, & u_1 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & a_2, & c_2, & u_2 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & a_3, & c_3, & u_3 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & a_4, & c_4, & u_4 \\ b_1, & b_2, & b_3, & b_4 & \cdots & \cdots & \cdots \\ c_1, & c_2, & c_3, & c_4 & \cdots & \cdots & \cdots \\ u_1, & u_2, & u_3, & u_4 & \cdots & \cdots & \cdots \end{vmatrix}.$$

For since this determinant must contain a constituent from each of the last three rows and columns it is of the first degree in u_{11}, &c., and the coefficient of any term u_{14} is

$$-\{\Sigma a_2c_3u_4\Sigma b_1c_2u_3+\Sigma a_1c_2u_3\Sigma b_2c_3u_4\} \text{ or } -(m_1n_4+m_4n_1).$$

In the determinant just written the matrix of the Hessian is bordered vertically with a, c, u; and horizontally with b, c, u. As we shall frequently have occasion to use determinants of this kind we shall find it convenient to denote them by an abbreviation, and shall write the result that we have just arrived at,

$$F+G=-2\begin{pmatrix}a, & c, & u\\ b, & c, & u\end{pmatrix}.$$

591. The quantity FG is transformed in like manner. It is evidently the product of

$$(m_1p_1+m_2p_2+m_3p_3+m_4p_4)\ (m_1q_1+m_2q_2+m_3q_3+m_4q_4),$$

and $\quad (n_1p_1+n_2p_2+n_3p_3+n_4p_4)\ (n_1q_1+n_2q_2+n_3q_3+n_4q_4).$

Now if the first line be multiplied out, and for every term $(p_1q_2+p_2q_1)$ we substitute its value derived from equation (I), it appears, as before, that the terms including t vanish, and it becomes $\Sigma\Sigma m_im_ju_{ij}$, which, as before, is equivalent to $\begin{pmatrix}a, & c, & u\\ a, & c, & u\end{pmatrix}$, where the notation indicates the determinant formed by bordering the matrix of the Hessian both vertically and horizontally with a, c, u. The second line is transformed in like manner; and we thus find that $(F+G)^3-3FG(F+G)=0$ transforms into

$$\begin{pmatrix}a, & c, & u\\ b, & c, & u\end{pmatrix}\left\{4\begin{pmatrix}a, & c, & u\\ b, & c, & u\end{pmatrix}^2-3\begin{pmatrix}a, & c, & u\\ a, & c, & u\end{pmatrix}\begin{pmatrix}b, & c, & u\\ b, & c, & u\end{pmatrix}\right\}=0.$$

It remains to complete the expansion of this symbolical expression, and to throw it into such a form that we may be able to divide out $c_1x_1+c_2x_2+c_3x_3+c_4x_4$. We shall for shortness write a, b, c, instead of $a_1x_1+a_2x_2+a_3x_3+a_4x_4$, b_1x_1+ &c., c_1x_1+ &c.

592. On inspection of the determinant, Art. 590, which we have called $\begin{pmatrix}a, & c, & u\\ b, & c, & u\end{pmatrix}$, it appears that since

$$u_{11}x_1+u_{12}x_2+u_{13}x_3+u_{14}x_4=(n-1)\ u_1,\ \text{\&c.},$$

this determinant may be reduced by multiplying the first four columns by x_1, x_2, x_3, x_4, and subtracting their sum from the last column multiplied by $(n-1)$, and similarly for the rows; when it becomes

$$\frac{1}{(n-1)^2}\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14}, & a_1, & c_1, & 0 \\ u_{21}, & u_{22}, & u_{23}, & u_{24}, & a_2, & c_2, & 0 \\ u_{31}, & u_{32}, & u_{33}, & u_{34}, & a_3, & c_3, & 0 \\ u_{41}, & u_{42}, & u_{43}, & u_{44}, & a_4, & c_4, & 0 \\ b_1, & b_2, & b_3, & b_4, & 0, & 0, & -b \\ c_1, & c_2, & c_3, & c_4, & 0, & 0, & -c \\ 0, & 0, & 0, & 0, & -a, & -c, & 0 \end{vmatrix},$$

which partially expanded is

$$-\frac{1}{(n-1)^2}\left\{c^2\binom{a}{b} - ac\binom{c}{b} - bc\binom{c}{a} + ab\binom{c}{c}\right\},$$

where $\binom{a}{b}$ denotes the matrix of the Hessian bordered with a single line, vertically of a's and horizontally of b's.

In like manner we have

$$\begin{pmatrix} a, & c, & u \\ a, & c, & u \end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c^2\binom{a}{a} - 2ac\binom{a}{c} + a^2\binom{c}{c}\right\},$$

$$\begin{pmatrix} b, & c, & u \\ b, & c, & u \end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c^2\binom{b}{b} - 2bc\binom{b}{c} + b^2\binom{c}{c}\right\}.$$

Now as it will be our first object to get rid of the letter a, we may make these expressions a little more compact by writing $cb_1 - bc_1 = d_1$, &c., when it is easy to see that

$$\binom{d}{d} = c^2\binom{b}{b} - 2bc\binom{b}{c} + b^2\binom{c}{c};$$

$$\binom{c}{d} = c\binom{b}{c} - b\binom{c}{c};\ \binom{a}{d} = c\binom{a}{b} - b\binom{a}{c}.$$

Thus

$$\begin{pmatrix} b, & c, & u \\ b, & c, & u \end{pmatrix} = -\frac{1}{(n-1)^2}\binom{d}{d},\ \begin{pmatrix} a, & c, & u \\ b, & c, & u \end{pmatrix} = -\frac{1}{(n-1)^2}\left\{c\binom{a}{d} - a\binom{c}{d}\right\},$$

and the equation of the surface, as given at the end of the last article, may be altered into

$$\left\{c\binom{a}{d} - a\binom{c}{d}\right\}\left[4\left\{c\binom{a}{d} - a\binom{c}{d}\right\}^2 - 3\binom{d}{d}\left\{c^2\binom{a}{a} - 2ac\binom{a}{c} + a^2\binom{c}{c}\right\}\right].$$

593. We proceed now to expand and substitute for each term $a_1a_2a_3$, &c., the corresponding differential coefficient. Then, in the first place, it is evident that

$$a^3 = n(n-1)(n-2)\,u = 0;\quad a^2a_1 = (n-1)(n-2)\,u_1, \text{ \&c.}$$

Hence $$a^2\begin{pmatrix}a\\c\end{pmatrix} = (n-1)(n-2)\begin{pmatrix}u\\c\end{pmatrix}.$$

But the last determinant is reduced, as in many similar cases, by subtracting the first four columns multiplied respectively by x_1, x_2, x_3, x_4 from the fifth column, and so causing it to vanish, except the last row. Thus we have

$$a^2\begin{pmatrix}a\\c\end{pmatrix} = -(n-2)\,Hc.$$

Again (see *Lessons on Higher Algebra*, Art. 36),

$$\begin{pmatrix}a\\a\end{pmatrix} = -\Sigma\frac{dH}{du_{mn}}a_ma_n.$$

We have therefore

$$a\begin{pmatrix}a\\a\end{pmatrix} = -(n-2)\,\Sigma\frac{dH}{du_{mn}}u_{mn} = -4(n-2)H.$$

Lastly, it is necessary to calculate $a\begin{pmatrix}a\\c\end{pmatrix}\begin{pmatrix}a\\d\end{pmatrix}$. Now if U_{mn} denote the minor obtained from the matrix of the Hessian by erasing the line and column which contain u_{mn}, it is easy to see that $a\begin{pmatrix}a\\c\end{pmatrix}\begin{pmatrix}a\\d\end{pmatrix} = -(n-2)\,\Sigma U_{mp}U_{qn}u_{mn}c_pd_q$, where the numbers m, n, p, q are each to receive in turn all the values 1, 2, 3, 4. But (see *Lessons on Higher Algebra*, Art. 33)

$$U_{mp}U_{nq} = U_{mn}U_{pq} - H\frac{dU_{pq}}{du_{mn}}.$$

Substituting this, and remembering that $\Sigma U_{mn}u_{mn} = 4H$, we have

$$a\begin{pmatrix}a\\c\end{pmatrix}\begin{pmatrix}a\\d\end{pmatrix} = -(n-2)\,H\begin{pmatrix}c\\d\end{pmatrix}.$$

Making then these substitutions we have

$$\left\{c\begin{pmatrix}a\\d\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix}\right\}^3 = c^3\begin{pmatrix}a\\d\end{pmatrix}^3 + 3(n-2)Hc^2\begin{pmatrix}c\\d\end{pmatrix}\begin{pmatrix}d\\d\end{pmatrix} - 3(n-2)Hcd\begin{pmatrix}c\\d\end{pmatrix}^2,$$

$$\left\{c\begin{pmatrix}a\\d\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix}\right\}\left\{c^2\begin{pmatrix}a\\a\end{pmatrix} - 2ac\begin{pmatrix}a\\c\end{pmatrix} + a^2\begin{pmatrix}c\\c\end{pmatrix}\right\}$$
$$= c^3\begin{pmatrix}a\\d\end{pmatrix}\begin{pmatrix}a\\a\end{pmatrix} + 4(n-2)Hc^2\begin{pmatrix}c\\d\end{pmatrix} - (n-2)Hcd\begin{pmatrix}c\\c\end{pmatrix}.$$

But attending to the meaning of the symbols d_1, &c., we see that d or $d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4$ vanishes identically. If then we substitute in the equation which we are reducing the values just obtained, it becomes divisible by c^3, and is then brought to the form

$$4\binom{a}{d}^3 - 3\binom{a}{d}\binom{a}{a}\binom{d}{d} = 0.$$

594. To simplify this further we put for d its value, when it becomes

$$4\left\{c\binom{b}{a} - b\binom{c}{a}\right\}^3 - 3\binom{a}{a}\left\{c\binom{b}{a} - b\binom{c}{a}\right\}\left\{c^2\binom{b}{b} - 2bc\binom{b}{c} + b^2\binom{c}{c}\right\}.$$

Now this is exactly the form reduced in the last article, except that we have b instead of a, and a instead of d. We can then write down

$$4\left\{c\binom{b}{a} - b\binom{c}{a}\right\}^3$$
$$= 4\left\{c^3\binom{b}{a}^3 + 3(n-2)Hc^2\binom{c}{a}\binom{a}{a} - 3(n-2)Hca\binom{c}{a}^2\right\},$$

while the remaining part of the equation becomes

$$3\binom{a}{a}\left\{c^3\binom{b}{a}\binom{b}{b} + 4(n-2)Hc^2\binom{c}{a} - (n-2)Hca\binom{c}{c}\right\}.$$

But (last article) the last term in both these can be reduced to $12(n-2)^2H^2c\binom{c}{c}$. Subtracting, then, the factor c^3 divides out again, and we have the final result cleared of irrelevant factors, expressed in the symbolical form

$$\binom{b}{a}\left\{4\binom{b}{a}^2 - 3\binom{b}{b}\binom{a}{a}\right\} = 0.$$

595. It remains to show how to express this result in the ordinary notation. In the first place we may transform it by the identity (see *Lessons on Higher Algebra*, Art. 33)

$$H\binom{a,\ b}{a,\ b} = \binom{a}{a}\binom{b}{b} - \binom{a}{b}^2,$$

whereby the equation becomes

$$\binom{b}{a}\binom{a}{a}\binom{b}{b} - 4H\binom{b}{a}\binom{a,\ b}{a,\ b} = 0.$$

Now $\binom{b}{a}\binom{a}{a}\binom{b}{b}$ expresses the covariant which we have before called Θ. For giving to U_{mn} the same meaning as before, the symbolical expression expanded may be written $\Sigma U_{mn} U_{pq} U_{rs} u_{mnr} u_{pqs}$, where each of the suffixes is to receive every value from 1 to 4. But the differential coefficient of H with respect to x_r can easily be seen to be $\Sigma U_{mn} u_{mnr}$, so that Θ is $\Sigma U_{rs} \frac{dH}{dx_r}\frac{dH}{dx_s}$, which is, in another notation, what we have called Θ, Art. 544. The covariant S is then reduced to the form $\Theta - 4H\Phi$, where

$$\Phi = \binom{b}{a}\binom{a,\ b}{a,\ b} = \Sigma U_{mn} U_{pq,rs} u_{mpq} u_{nrs},$$

where $U_{pq,rs}$ denotes a second minor formed by erasing two rows and two columns from the matrix of the Hessian, a form scarcely so convenient for calculation as that in which I had written the equation, *Philosophical Transactions*, CL. 1860, p. 239. For surfaces of the third degree Clebsch has observed that Φ reduces, as was mentioned before, to $\Sigma U_{mn} H_{mn}$, where H_{mn} denotes a second differential coefficient of H.

596. *The surface S touches the surface H along a certain curve.* Since the equation S is of the form $\Theta - 4H\Phi = 0$, it is sufficient to prove that Θ touches H. But since Θ is got by bordering the matrix of the Hessian with the differentials of the Hessian, $\Theta = 0$ is equivalent to the symbolical expression $\binom{H}{H} = 0$. But, by an identical equation already made use of, we have

$$H\binom{c,\ H}{c,\ H} = \binom{H}{H}\binom{c}{c} - \binom{H}{c}^2,$$

where c is arbitrary. Hence Θ touches H along its intersection with the surface $\binom{H}{c}$ of degree $7n - 15$. It is proved then that S touches H, and that through the curve of contact an infinity of surfaces can pass of degree $7n - 15$.

597. The equation of the surface generated by the flecnodal tangents is got by eliminating $x'y'z'w'$ between $U'=0$, $\Delta U'=0$, $\Delta^2U'=0$, $\Delta^3U'-0$; which result, by the ordinary rule, is of degree,

$$n(n-2)(n-3)+2n(n-1)(n-3)+3n(n-1)(n-2)$$
$$=6n^3-22n^2+18n.$$

Now this result expresses the locus of points, whose first, second, and third polars intersect on the surface; and, since if a point be anywhere on the surface, its first, second, and third polars intersect in six points on the surface, we infer that the result of elimination must be of the form $U^6M=0$. The degree of M, which is the scroll (A), is therefore

$$a=2n(n-3)(3n-2).$$

598. We can in like manner solve problem (B) of Art. 588. For the point of contact of an inflexional tangent we have $U'=0$, $\Delta U'=0$, $\Delta^2U'=0$; and if it touch the surface again, we have also $W'=0$, where W' is the discriminant of the equation of degree $n-3$ in $\lambda:\mu$, which remains when the first three terms of the equation of Art. 272 vanish. For W' then we have $\lambda''=(n+3)(n-4)$, $\mu''=(n-3)(n-4)$; and having, as in Art. 588 and last article, $\lambda=1$, $\mu=n-1$; $\lambda'=2$, $\mu'=n-2$, we find for the degree of the resultant .

$$2(n-3)(n-4)+(n-2)(n+3)(n-4)$$
$$+2(n-1)(n+3)(n-4)-2(n+3)(n-4)$$
$$=(n-4)(3n^2+5n-24).$$

The locus of the points of three-point contact of inflexional tangents which touch the surface elsewhere is therefore of degree

$$b_3=n(n-4)(3n^2+5n-24).$$

In order that a tangent at the point $x'y'z'w'$ may elsewhere be an inflexional tangent, we must have $\Delta U'=0$ (an equation for which $\lambda=1$, $\mu=n-1$), and, besides, we must have satisfied the system of two conditions, that the equation of degree $n-2$ in $\lambda:\mu$, which remains when the first two terms vanish of the equation of Art. 272, may have three roots all equal to each other. If then λ', μ'; λ'', μ'' be the

degrees in which the variables enter into these two conditions, the degree of the surface which passes through the points of simple contact is, by Art. 473, $\lambda'\mu''+\lambda''\mu'+(n-2)\lambda'\lambda''$. But (see *Higher Algebra*, Lesson 19)

$$\lambda'\lambda''=(n-4)\ (n^2+n+6),\ \lambda'\mu''+\lambda''\mu'=(n-2)\ (n-4)\ (n+6);$$

the degree of the resultant is $(n-2)\ (n-4)\ (n^2+2n+12)$, and the locus of the points of simple contact of tangents which are inflexional elsewhere is of degree

$$b_2=n\ (n-2)\ (n-4)\ (n^2+2n-12).$$

The equation of the scroll (B) generated by the tangents is found by eliminating $x'y'z'w'$ between the four equations $U'=0$, $\Delta U'=0$, $\Delta^2U'=0$, $W'=0$; and from what has been stated as to the degree of the variables in each of these equations the degree of the resultant is

$$n\ (n-2)\ (n-3)\ (n-4)+2n\ (n-1)\ (n-3)\ (n-4)$$
$$+n\ (n-1)(n-2)(n+3)\ (n-4)=n\ (n-4)\ (n^3+3n^2-20n+18).$$

But it appears, as in the last article, that this resultant contains as a factor, U in the power $2\ (n+3)\ (n-4)$. Dividing out this factor the degree of the scroll (B) remains

$$b=n\ (n-3)\ (n-4)\ (n^2+6n-4).$$

599. Next we have the problem (C). The locus of the points of contact of triple tangent lines is investigated in like manner, except that for the conditions that the equation considered above should have three roots all equal, we substitute the conditions that the same equation should have two distinct pairs of equal roots. But (see *Higher Algebra*, Lesson 19) for this system of conditions we have

$$\lambda'\lambda''=\tfrac{1}{2}\ (n-4)\ (n-5)\ (n^2+3n+6),$$
$$\lambda'\mu''+\lambda''\mu'=(n-2)\ (n-4)\ (n-5)\ (n+3).$$

The degree of the resultant is, therefore,

$$\tfrac{1}{2}\ (n-2)\ (n-4)\ (n-5)\ (n^2+5n+12),$$

and the degree of the locus of the points of contact of triple tangents is

$$c_2=\tfrac{1}{2}n\ (n-2)\ (n-4)\ (n-5)\ (n^2+5n+12).$$

To find the scroll (C) generated by the triple tangents, we are to eliminate $x'y'z'w'$ between $U'=0$, $\Delta U'=0$, and the two

conditions, the degree of the resultant being

$$n\mu'\mu'' + n(n-1)(\lambda'\mu'' + \lambda''\mu');$$

but since this resultant contains as a factor $U^{\lambda'\lambda''}$, we have to subtract $n\lambda'\lambda''$ from the number just written. Substituting $\frac{1}{2}(n-2)(n-3)(n-4)(n-5)$ for $\mu'\mu''$, and the values last given for $\lambda'\lambda''$, $\lambda'\mu'' + \lambda''\mu'$, we get, for the degree of the scroll (C), after dividing by three,

$$c = \tfrac{1}{3}n(n-3)(n-4)(n-5)(n^2+3n-2).$$

The following examples are solved by the numbers found in Art. 588 and the last three articles :—

Ex. 1. To find the degree of the curve formed by the points of simple intersection of the four-point tangents.

The complete curve of intersection with U of the ruled surface M whose degree is a consists of the curve of points of simple intersection, whose degree we call a_1, and of the curve of fourfold points, of degree a_4. We have manifestly $4a_4 + a_1 = na$. Putting in the values of a, a_4, we find

$$a_1 = 2n(n-4)(3n^2+n-12).$$

Ex. 2. To find the degree of the curve formed by the points of simple intersection of inflexional tangents which touch the surface again.

The complete curve of intersection of the scroll (B) with U consists of the curve of points at which the tangents are inflexional, of degree b_3; of that of the ordinary contacts, of degree b_2; and of that of the simple intersections, of degree b_1. Among these we have the obvious relation $nb = 3b_3 + 2b_2 + b_1$; putting in the values of b, b_3, b_2 we find

$$b_1 = n(n-4)(n-5)(n^3+6n^2-n-24).$$

Ex. 3. To find the degree of the curve formed by the points of simple intersection of triple ordinary tangent lines.

Here with a similar notation $nc = 2c_2 + c_1$, whence we have

$$c_1 = \tfrac{1}{3}n(n-4)(n-5)(n-6)(n^3+3n^2-2n-12).$$

600. There remains to be considered another class of problems, the determination of the number of tangents which satisfy four conditions. The following is an enumeration of these problems. To determine the number of lines which are :

(β) five-point tangents;

(γ) flecnodal (four-point) tangents in one place and ordinary tangents in another;

(δ) inflexional (three-point) in two places;

(ϵ) inflexional in one place and ordinary tangents in two others;

(ζ) ordinary tangents in four places.

601. To find the points on a surface where a line can be drawn to meet the surface in five consecutive points, we have to form the condition that the intersection of $\Delta U'$, $\Delta^2 U'$, and an arbitrary plane should satisfy $\Delta^4 U'$ as well as $\Delta^3 U'$. Clebsch applied to $\Delta^4 U'$ the same symbolical method of elimination which has been already applied to $\Delta^3 U'$. He succeeded in dividing out the factor c^6 from this result; but in the final form which he found, and for which I refer to his memoir, there remain c symbols in the second degree, and the result being of degree $14n - 30$ in the variables, all that can be concluded from it is that through the points which I have called β (Art. 600) an infinity of surfaces can be drawn of degree $14n - 30$. We can say, therefore, that the number of such points does not exceed $n(11n - 24)(14n - 30)$.

602. The numerical solutions of the problems proposed in Art. 600 accomplished by Dr. Schubert * are derived from the principle of correspondence, which may be stated as follows:

Take any line and consider the correspondence between two planes through it, such that when the first passes through a given point there are p points which determine the second, and when the second passes through a given point, q points determine the first, and, moreover, such that there are g pairs of corresponding points whose connecting lines meet an arbitrary right line, then the number of planes of the system which contain a pair of corresponding points is $p + q$; but since of these there are g whose connecting lines meet the arbitrary line, the remaining $p + q - g$ contain coinciding pairs of points of the systems.

603. Now to determine β. A five-point contact arises from a four-point contact by the coincidence of one additional

* *Gött. Nachr.*, Feb. 1876; *Math. Ann.*, x. p. 102, xi. pp. 348-78. See also his *Kalkül der abzählenden Geometrie* (1879), pp. 236-7, 246.

simple point of intersection. To each of the a_4 points in a plane correspond $n-4$ simple intersections of the osculating tangents at them with U; and to each of the points a_1 in the plane corresponds a single fourfold point. Hence the number $p+q$ for these two systems is $(n-4)\ a_4+a_1$. But the surface M meets any right line in a points through each of which passes a line connecting the $n-4$ points a_1 to the corresponding a_4; hence in this case g is $(n-4)\ a$. Accordingly, the number of coincidences of a point a_1 with a point a_4 is

$$\beta=(n-4)\ a_4+a_1-(n-4)\ a=(n-8)\ a_4+4a$$
$$=5n\ (n-4)\ (7n-12).$$

The same number is found from the analogous relation

$$\beta=b_2+b_3-b,$$

since the union of a three-point with an ordinary contact also leads to a five-point one.

Again, four-point tangents having another ordinary contact may arise either through coincidence of two simple intersections on a four-point tangent, giving in a similar manner by the principle of correspondence

$$\gamma=2\ (n-5)\ a_1-(n-5)\ (n-4)\ a;$$

or, through the coincidence of a simple intersection with the three-point contact of an inflexional tangent which touches elsewhere, giving

$$\gamma=(n-5)\ b_3+b_1-(n-5)\ b;$$

or, lastly, by the coincidence of two contacts of a triple ordinary tangent, giving

$$\gamma=4c_2-6c.$$

Each method leads to

$$\gamma=2n\ (n-4)\ (n-5)\ (3n-5)\cdot(n+6).$$

Tangents inflexional in two places arise from the coincidences of an ordinary intersection with an ordinary contact on an inflexional tangent, thus

$$(n-5)\ b_2+b_1-(n-5)\ b=2\delta,$$

which gives

$$\delta=\tfrac{1}{2}n\ (n-4)\ (n-5)\ (n^3+3n^2+29n-60).$$

Inflexional tangents having two further ordinary contacts

arise from coincidences of two simple intersections among those on inflexional tangents having one other ordinary contact, thus

$$2\epsilon = 2(n-6)\,b_1 - (n-5)(n-6)\,b\,;$$

or, from coincidence of a simple intersection with one of the ordinary contacts among those on tangents having three such, whence

$$\epsilon = (n-6)\,c_2 + 3c_1 - 3(n-6)\,c$$
$$= \tfrac{1}{2}n\,(n-4)(n-5)(n-6)(n^3+9n^2+20n-60).$$

Finally, four ordinary contacts arise from coincidence of two simple intersections in the case of a tangent line having three ordinary contacts. Whence

$$4\zeta = 2(n-7)\,c_1 - (n-6)(n-7)\,c\,;$$
$$\zeta = \tfrac{1}{12}n\,(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30).$$

SECTION IV.—CONTACT OF PLANES WITH SURFACES.

604. We can discuss the cases of planes which touch a surface in the same algebraic manner as we have done those of touching lines. Every plane which touches a surface meets it in a section having a double point; but since the equation of a plane includes three constants, a determinate number of tangent planes can be found which will fulfil two additional conditions. And if but one additional condition be given, an infinite series of tangent planes can be found which will satisfy it, those planes enveloping a developable, and their points of contact tracing out a curve on the surface. It may be required either to determine the number of solutions when two additional conditions are given, or to determine the nature of the curves and developables just mentioned, when one additional condition is given.

[The former class of problem determines the six types of singular tangent planes; the following are the singularities of the sections and the numbers of the planes: *

[* For a full discussion of these the reader is referred to the original memoirs: Salmon, *Trans. R. Ir. Ac.*, XXIII. p. 461, 1855; Clebsch, *Crelle*, LXIII. p. 14, 1863; Cayley, *Coll. Math. Papers*, VI. p. 359, 1869; Schubert, *Math. Ann.*, XI. p. 375, 1877; Basset, *Quart. J.*, XL. p. 210, 1908; XLI. p. 21, 1909.]

(i) a biflecnode,

$\hat{\omega}_4 = a = 5n\ (7n^2 - 28n + 30)$;

(ii) a node with an undulation on one branch,

$\hat{\omega}_0 = \beta = 5n\ (n - 4)\ (7n - 12)$;

(iii) a tacnode,

$\hat{\omega}_5 = 2n\ (n - 2)\ (11n - 24)$;

(iv) a flecnode and another node,

$\hat{\omega}_2 = n\ (n - 2)\ (11n - 24)\ (n^3 - n^2 + n - 16)$;

(v) a cusp and a node,

$\hat{\omega}_1 = 4n\ (n - 2)\ (n - 3)\ (n^3 + 3n - 16)$;

(vi) three nodes,

$$\hat{\omega}_3 = \tfrac{1}{6}n\ (n - 2)\ (n^7 - 4n^6 + 7n^5 - 45n^4 + 114n^3 - 111n^2 + 548n - 960).]$$

The first of these problems has been solved, as follows, by Clebsch, but with an erroneous result, as has been shown by Schubert. It was proved (Art. 537) that the points of inflexion of the section by the tangent plane at any point on a surface, of the polar cubic of that point, lie on the plane $xH_1 + yH_2 + zH_3 + wH_4$. Let it be required now to find the locus of points $x'y'z'w'$ on a surface such that the line joining $x'y'z'w'$ to one of these points of inflexion may meet any assumed line: this is, in other words, to find the condition that coordinates of the form $\lambda x' + \mu x$, $\lambda y' + \mu y$, &c. (where $xyzw$ is the intersection of the assumed line with the tangent plane) may satisfy the equation of the polar with respect to the Hessian $\varDelta H'$, and also of the polar cubic $\varDelta^3 U'$. Now the result of substitution in $\varDelta H'$ is $4\ (n - 2)\ \lambda H' + \mu \varDelta H'$. When we substitute in $\varDelta^3 U'$, the coefficient of λ^3 vanishes because $x'y'z'w'$ is on the surface, and that of λ^2 vanishes because $xyzw$ is in the tangent plane. The result is then $3\ (n - 2)\ \lambda \varDelta^2 U' + \mu \varDelta^3 U' = 0$. Eliminating $\lambda : \mu$ between these two equations, we have $4H'\varDelta^3 U' = 3\varDelta H' \varDelta^2 U'$, where in $\varDelta^3 U'$, &c., we are to substitute the coordinates of the intersection of an arbitrary line with the tangent plane; that is to say, the several determinants of the system

$$\left\| \begin{matrix} u_1, & u_2, & u_3, & u_4 \\ a, & \beta, & \gamma, & \delta \\ a', & \beta', & \gamma', & \delta' \end{matrix} \right\| .$$

By this substitution $\Delta^3 U'$ becomes in $x'y'z'w'$ of degree $n-3+3(n-1)=4n-6$, and H' being of degree $4(n-2)$, the equation is of degree $8n-14$. This, then, is the degree of the locus required.

Now the points at which two four-point tangents can be drawn belong to this locus. At any one of these points the four-point tangents evidently both lie on the polar cubic of that point, and their plane will therefore intersect that cubic in a third line which, as we saw (Art. 537), lies in the plane $\Delta H'$. *Every* point on that line is to be considered as a point of inflexion of the polar cubic; and therefore the plane through the point $x'y'z'w'$ and any arbitrary line *must* pass through a point of inflexion. The points then, whose number we are investigating, and which are evidently double points on the curve US, are counted doubly among the $n(11n-24)(8n-14)$ intersections of the curve US with the locus determined in this article. Let us examine now what other points of the curve US can belong to the locus. At any point on this curve the four-point tangent lies in the polar cubic, the section of which by the tangent plane consists of this line and a conic; and since all the points of inflexion of such a system lie in the line, the four-point tangent itself is, in this case, the only line joining $x'y'z'w'$ to a point of inflexion. And we have seen (Art. 597) that the number of such tangents which can meet an assumed line is $2n(n-3)(3n-2)$. Now Schubert first pointed out in applying his method of enumeration to the present problem, as we shall immediately show, that these lines must be counted three times. We have, then, the equation

$$2a+6n(n-3)(3n-2)=n(11n-24)(8n-14),$$

whence $$a=5n(7n^2-28n+30),$$

which is the solution of the problem proposed. Schubert also deduces this result from the principle of correspondence.

The points of contact of the inflexional tangents which meet an arbitrary given right line l are easily shown as in Art. 576, Ex. 2, to lie on the intersection of U with a surface of degree $3n-4$. This surface meets the flecnodal curve

(see notation in Examples, Art. 599) in $(3n-4)\ a_4$ points, which consist of the a points of contact of four-point tangents which meet the line l, and the $d=(3n-4)\ a_4 - a$ flecnodes, whose ordinary inflexional tangent meets l.

Accordingly, we may suppose a pencil of rays in a plane such that to each ray which meets a four-point tangent corresponds one which meets the other inflexional tangent at the same flecnode. In such a pencil there will be $a+d=(3n-4)\ a_4$ rays meeting as well a four-point tangent as also the other inflexional tangent at its flecnode. But these rays include the a_4 rays to the points of the flecnodal curve in the plane of the pencil and $(n-1)\ a_4$ which lie in the tangent planes through the vertex of the pencil to U at flecnodes. Thus there remain

$$a+d-a_4-(n-1)\ a_4 = 2\ (n-2)\ a_4$$

rays having the above property. These must be the rays which intersect tangents which have fourfold contact at parabolic points. It is not difficult to show otherwise from Art. 596 by the usual algebraical methods that there are

$$2n\ (n-2)\ (11n-24)$$

points on a surface of the degree n in which *coincident inflexional tangents have a four-point contact.*

The d tangent lines generate a ruled surface intersecting U in a curve of degree nd which consists of the curve of threefold points whose degree is a_4 and of that of ordinary intersections of degree a_1'. These give

$$a_1' + 3a_4 = nd.$$

Now applying the principle of correspondence, to each of the a_4 points in a plane correspond $n-3$ simple intersections of the tangents at them with U and to each of the points a_1' corresponds a single flecnode. But the surface generated by d lines meets any right line in d points through each of which pass $n-3$ lines connecting a point a_1' with a point a_4. Hence putting $(n-3)\ d$ for g,

$$a_1' + (n-3)\ a_4 - (n-3)\ d$$

is the number of coincidences of a flecnode and one of the simple points on the ordinary inflexional tangent. Now we

saw that in $2(n-2)\,a_4$ fourfold points the two osculating tangents coincide, hence the difference

$$a_1' + (n-3)\,a_4 - (n-3)\,d - 2\,(n-2)\,a_4 = (8n-14)\,a_4 - 3a$$

is double the number of biflecnodal points, as above.

605. The second class of problems, referred to at the beginning of Art. 604, determines the curves traced out by the points of contact of, and the torses enveloped by, the tangent planes which meet the surfaces in sections having (i) a flecnode, (ii) a cusp, (iii) two nodes. The first has been partly considered by anticipation in Art. 588; the other two we shall now consider.

Let the coordinates of three points be $x'y'z'w'$, $x''y''z''w''$, $xyzw$; then those of any point on the plane through the points will be $\lambda x' + \mu x'' + \nu x$, $\lambda y' + \mu y'' + \nu y$, &c.; and if we substitute these values for $xyzw$ in the equation of the surface, we shall have the relation which must be satisfied for every point where this plane meets the surface. Let the result of substitution be $[U]=0$, then $[U]$ may be written

$$\lambda^n U' + \lambda^{n-1}\mu\Delta_{,,}U' + \lambda^{n-1}\nu\Delta U' + \tfrac{1}{2}\lambda^{n-2}(\mu\Delta_{,,} + \nu\Delta)^2 U' + \&c. = 0,$$

where
$$\Delta_{,,} = x''\frac{d}{dx'} + y''\frac{d}{dy'} + z''\frac{d}{dz'} + w''\frac{d}{dw'},$$

$$\Delta = x\frac{d}{dx'} + y\frac{d}{dy'} + z\frac{d}{dz'} + w\frac{d}{dw'}.$$

The plane will touch the surface if the discriminant of this equation in λ, μ, ν vanish. If we suppose two of the points fixed and the third to be variable, then this discriminant will represent all the tangent planes to the surface which can be drawn through the line joining the two fixed points.

We shall suppose the point $x'y'z'w'$ to be on the surface, and the point $x''y''z''w''$ to be taken anywhere on the tangent plane at that point; then we shall have $U'=0$, $\Delta_{,,}U'=0$, and the discriminant will become divisible by the square of $\Delta U'$. For of the tangent planes which can be drawn to a surface through any tangent line to that surface, two will coincide with the tangent plane at the point of contact of

that line. If the tangent plane at $x'y'z'w'$ be a double tangent plane, then the discriminant we are considering, instead of being, as in other cases, only divisible by the square of the equation of the tangent plane, will contain its cube as a factor. In order to examine the condition that this may be so, let us, for brevity, write the equation $[U]$ as follows, the coefficients of λ^n, $\lambda^{n-1}\mu$ being supposed to vanish,

$$T\lambda^{n-1}\nu + \tfrac{1}{2}\lambda^{n-2}(A\mu^2 + 2B\mu\nu + C\nu^2) + \&c. = 0.$$

T represents the tangent plane at the point we are considering, C its polar quadric, while $A = 0$ is the condition that $x''y''z''w''$ should lie on that polar quadric. Now it will be found that the discriminant of $[U]$ is of the form

$$T^2A\ (B^2 - AC)^2\phi + T^3\psi = 0,$$

where ϕ is the discriminant when T vanishes as well as U' and $\Delta_{\prime\prime}U'$. In order that the discriminant may be divisible by T^3, some one of the factors which multiply T^2 must either vanish or be divisible by T.

606. First, then, let A vanish. This only denotes that the point $x''y''z''w''$ lies on the polar quadric of $x'y'z'w'$; or, since it also lies in the tangent plane, that the point $x''y''z''w''$ lies on one of the inflexional tangents at $x'y'z'w'$. Thus we learn that if the class of a surface be p, then of the p tangent planes which can be drawn through an ordinary tangent line, two coincide with the tangent plane at its point of contact, and there can be drawn $p - 2$ distinct from that plane; but that if the line be an inflexional tangent, three will coincide with that tangent plane, and there can be drawn only $p - 3$ distinct from it. If we suppose that $x''y''z''w''$ has not been taken on an inflexional tangent, A will not vanish, and we may set this factor aside as irrelevant to the present discussion.

We may examine, at the same time, the conditions that T should be a factor in $B^2 - AC$, and in ϕ.

The problem which arises in both these cases is the following: Suppose that we are given a function V, whose degrees in $x'y'z'w'$, in $x''y''z''w''$ and in $xyzw$ are respectively

λ, μ, μ. Suppose that this represents a surface, having as a multiple line of order μ, the line joining the first two points; or, in other words, that it represents a series of planes through that line; to find the condition that one of these planes should be the tangent plane T, whose degrees are $n-1$, 0, 1. If so, any arbitrary line which meets T will meet V, and therefore if we eliminate between the equations $T=0$, $V=0$, and the equations of an arbitrary line

$$ax+by+cz+dw=0,\ a'x+b'y+c'z+d'w=0,$$

the resultant R must vanish. This is of degree μ in $abcd$, in $a'b'c'd'$, and in $x''y''z''w''$, and of degree $\mu(n-1)+\lambda$ in $x'y'z'w'$. But evidently if the assumed right line met the line joining $x'y'z'w'$, $x''y''z''w''$, R would vanish even though T were not a factor in V. The condition ($M=0$), that the two lines should meet, is of the first degree in all the quantities we are considering; and we see now that R is of the form $M^\mu R'$. R' remains a function of $x'y'z'w'$ alone, and is of degree $\mu\ (n-2)+\lambda$.

607. To apply this to the case we are considering, since the discriminant of $[U]$ represents a series of planes through $x'y'z'w'$, $x''y''z''w''$, it follows that B^2-AC and ϕ both represent planes through the same line. The first is of the degrees $2\ (n-2)$, 2, 2, while ϕ is of the degrees $(n-2)\ (n^2-6)$, n^3-2n^2+n-6, n^3-2n^2+n-6, as appears by subtracting the sum of the degrees of T^2, A, and $(B^2-AC)^2$ from the degrees of the discriminant of $[U]$, which is of degree $n\ (n-1)^2$ in all the variables. It follows then from the last article that the condition ($H=0$) that T should be a factor in B^2-AC is of degree $4\ (n-2)$, and the condition ($K=0$) that T should be a factor in ϕ is of degree $(n-2)\ (n^3-n^2+n-12)$. At all points then of the intersection of U and H the tangent plane must be considered double. H is no other than the Hessian; the tangent plane at every point of the curve UH meets the surface in a section having a cusp, and is to be counted as double (Art. 269). The curve UK is the locus of points of contact of planes which touch the surface in two distinct

points (Art. 286). It is called by Prof. Cayley the node-couple curve.

608. Let us consider next the series of tangent planes which touch the surface along the curve UH. They form a developable whose degree is $2n(n-2)(3n-4)$ (Art. 576, Ex. 3). The class of the same developable, or the number of planes of the system which can be drawn through an assigned point, is $4n(n-1)(n-2)$. For the points of contact are evidently the intersections of the curve UH with the first polar of the assigned point. We can also determine the number of stationary planes of the system. If the equation of U, the plane z being the tangent plane at any point on the curve UH, be $z+y^2+u_3+\&c.=0$, it is easy to show that the direction of the tangent to UH is in the line $\frac{d^2u_3}{dx^2}=0$. Now the tangent planes to U are the same at two consecutive points proceeding along the inflexional tangent y. If then u_3 do not contain any term x^3 (that is to say, if the inflexional tangent meet the surface in four consecutive points), the direction of the tangent to the curve UH is the same as that of the inflexional tangent; and the tangent planes at two consecutive points on the curve UH will be the same. The number of stationary tangent planes is then equal to the number of intersections of the curve UH with the surface S. But since the curve touches the surface (Art. 596), we have

$$\alpha = 2n(n-2)(11n-24).$$

From these data all the singularities of the developable which touches the surface along UH can be determined as in Art. 326; we have

$$\text{degree} = n(n-2)(28n-60),\quad \text{class} = 4n(n-1)(n-2),$$
$$\text{rank} = 2n(n-2)(3n-4),$$
$$\alpha = 2n(n-2)(11n-24),\quad \beta = n(n-2)(70n-160);$$
$$2g = n(n-2)(16n^4-64n^3+80n^2-108n+156),$$
$$2h = n(n-2)(784n^4-4928n^3+10320n^2-7444n+548).$$

The developable here considered answers to a cuspidal line on the reciprocal surface, whose singularities are got by

interchanging degree and class, α and β, &c., in the above formulæ.

The class of the developable touching the surface along UK, which is the degree of a double curve on the reciprocal surface, is seen as above to be $n(n-1)(n-2)(n^3-n^2+n-12)$. Its other singularities will be obtained in the next Chapter, where we shall also determine the number of solutions in some cases where a tangent plane is required to fulfil two other conditions.

CHAPTER XVII*a*.

THEORY OF RECIPROCAL SURFACES.

609. [We have seen that a surface without multiple points has in general two sets of singular tangent planes, viz. the node-couple torse and the spinode torse, while the reciprocal surface has a double curve and a cuspidal curve. These singularities, which must occur either on the surface or on its reciprocal, are called *ordinary* singularities; in dealing with the general theory of reciprocals, we suppose the original surface to possess them all.

In connexion with these, there present themselves finite numbers of other singular points and planes, which will, however, be different according as the surface is regarded as a locus or an envelope. For example, on the double curve there are points of transition between the parts where the section of the surface has a node with real and imaginary tangents respectively. For an envelope, these are points of intersection of the double and cuspidal curves which are stationary on the latter (see Art. 624); while for a locus, they are the pinch-points, at which the two tangent planes coincide. Further, each of these singularities is in general accompanied by others; for example, at a pinch-point, any section of the surface has a cusp, but certain sections have higher singularities, and their planes are singular tangent planes.]

Let b, c be the degrees of the double and cuspidal curves. The tangent cone, determined as in Art. 277, includes doubly the cone standing on the double curve and trebly that standing on the cuspidal curve, so that if the degree of the tangent cone proper be a, we have

$$a + 2b + 3c = n\,(n-1).$$

The class of the cone is the same as the degree of the reciprocal. Let the cone have δ double and κ cuspidal edges. Let b have k apparent double points, and t triple points which are also triple points on the surface; and let c have h apparent double points. Let the curves b and c intersect in γ points which are stationary points on the former, in β which are stationary points on the latter, and in i which are singular points on neither. Let the curve of contact a meet b in ρ points, and c in σ points. Let the same letters accented denote singularities of the reciprocal surface.

610. We saw (Art. 279) that the points where the curve of contact meets $\Delta^2 U$, give rise to cuspidal edges on the tangent cone. But when the line of contact consists of the complex curve $a+2b+3c$, and when we want to determine the number of cuspidal edges on the cone a, the points where b and c meet $\Delta^2 U$ are plainly irrelevant to the question. Neither shall we have cuspidal edges answering to all the points where a meets $\Delta^2 U$, since a common edge of the cones a and c is to be regarded as a cuspidal edge of the complex cone, although not so on either cone considered separately. The following formulæ contain an analysis of the intersections of each of the curves a, b, c, with the surface $\Delta^2 U$,

$$\left.\begin{aligned} a\ (n-2) &= \kappa+\rho+2\sigma \\ b\ (n-2) &= \rho+2\beta+3\gamma+3t \\ c\ (n-2) &= 2\sigma+4\beta+\gamma \end{aligned}\right\}\ldots\ldots\ldots\ldots(A).$$

The reader can see without difficulty that the points indicated in these formulæ are included in the intersections of $\Delta^2 U$ with a, b, c, respectively; but it is not so easy to see the reason for the numerical multipliers which are used in the formulæ. Although it is probably not impossible to account for these constants by *a priori* reasoning, I prefer to explain the method by which I was led to them inductively.*

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* The first attempt to explain the effect of nodal and cuspidal lines on the degree of the reciprocal surface was made in the year 1847, in two papers which I contributed to the *Cambridge and Dublin Mathematical Journal*, II. p. 65, and IV. p. 188. It was not till the close of the year 1849,

611. We know that the reciprocal of a cubic is a surface of the twelfth degree, which has a cuspidal edge of the twenty-fourth degree, since its equation is of the form $64S^3 = T^2$, where S is of the fourth and T of the sixth degree (Art. 519). Each of the twenty-seven lines (Art. 530) on the surface answers to a double line on the reciprocal. The proper tangent cone, being the reciprocal of a plane section of the cubic, is of the sixth degree, and has nine cuspidal edges. Thus we have $a' = 6$, $b' = 27$, $c' = 24$, $n' = 12$, $a' + 2b' + 3c' = 12.11$. The intersections of the curves c' and b' with the line of contact of a cone a' through any assumed point, answer to tangent planes to the original cubic, whose points of contact are the intersections of an assumed plane with the parabolic curve UH, and with the twenty-seven lines. Consequently there are twelve points σ' and twenty-seven points ρ'; one of the latter points lying on each of the lines, of which the double line of the reciprocal surface is made up.

Now the sixty points of intersection of the curve a' with the second polar, which is of the tenth degree, consist of the nine points κ', the twenty-seven points ρ', and the twelve points σ'. It is manifest, then, that the last points must count double, since we cannot satisfy an equation of the form $9\lambda + 27\mu + 12\nu = 60$, by any integer values of λ, μ, ν, except (1, 1, 2). Thus we are led to the first of the equations (A).

Consider now the points where any of the twenty-seven lines b meet the same surface of the tenth degree. The points β' answer to the points where the twenty-seven right lines touch the parabolic curve; and there are two such points on each of these lines (Art. 287). There are also five points t on each of these lines (Art. 530), and we have just seen that there is one point ρ. Now, since the equation $\lambda + 2\mu + 5\nu = 10$, can have only the systems of integer solutions (1, 2, 1) or

however, that the discovery of the twenty-seven right lines on a cubic by enabling me to form a clear conception of the nature of the reciprocal of a cubic, led me to the theory in the form here explained. Some few additional details will be found in a memoir which I contributed to the *Transactions of the Royal Irish Academy*, XXIII. p. 461.

(3, 1, 1), the ten points of intersection of one of the lines with the second polar must be made up either $\rho' + 2\beta' + t'$, or $3\rho' + \beta' + t'$, and the latter form is manifestly to be rejected. But, considering the curve b' as made up of the twenty-seven lines, the points t' occur each on three of these lines: we are then led to the formula $b'(n' - 2) = \rho' + 2\beta' + 3t'$.

The example we are considering does not enable us to determine the coefficient of γ in the second formula (A), because there are no points γ on the reciprocal of a cubic.

Lastly, the two hundred and forty points in which the curve c meets the second polar are made up of the twelve points σ', and the fifty-four points β'. Now the equation $12\lambda + 54\mu = 240$ only admits of the systems of integer solutions (11, 2), or (2, 4), and the latter is manifestly to be preferred. In this way we are led to assign all the coefficients of the equations (A) except those of γ.

612. Let us now examine in the same way the reciprocal of a surface of degree n, which has no multiple points. We have then $n' = n(n-1)^2$, $n' - 2 = (n-2)(n^2+1)$, $a' = n(n-1)$; and for the double and cuspidal curves we have (Art. 286) $b' = \frac{1}{2}n(n-1)(n-2)(n^3 - n^2 + n - 12)$, $c' = 4n(n-1)(n-2)$. The number of cuspidal edges on the tangent cone to the reciprocal, answering to the number of points of inflexion on a plane section of the original, gives us $\kappa' = 3n(n-2)$. The points ρ' and σ' answer to the points of intersection of an assumed plane with the curves UK and UH (Art. 607); hence $\rho' = n(n-2)(n^3 - n^2 + n - 12)$, $\sigma' = 4n(n-2)$. Substitute these values in the formula $a'(n'-2) = \kappa' + \rho' + 2\sigma'$, and it is satisfied identically, thus verifying the first of formulæ (A).

We shall next apply the same case to the third of the formulæ (A). It was proved (Art. 608) that the number of points β' is $2n(n-2)(11n-24)$. Now the intersections of the double and cuspidal curves on the reciprocal surface answer to the planes which touch the original surface at the points of meeting of the curves UH and UK. If a plane meet the surface in a section having an ordinary double

point and a cusp, since from the mere fact of its touching at the latter point it is a double tangent plane, it belongs in two ways to the system which touches along UK; or, in other words, it is a stationary plane of that system. And, since evidently the points β' are to be included in the intersections of the double and cuspidal curves, the points U, H, K must either answer to points β' or points γ'. Assuming, as it is natural to do, that the points β' count double among the intersections of UHK, we have

$$\gamma' = n\{4(n-2)\}\{(n-2)(n^3-n^2+n-12)\} - 4n(n-2)(11n-24)$$
$$= 4n(n-2)(n-3)(n^3+3n-16).$$

But if we substitute the values already found for c', n', σ', β', the quantity $c'(n'-2) - 2\sigma' - 4\beta'$ becomes also equal to the value just assigned for γ'. Thus the third of the formulæ (A) is verified. It would have been sufficient to assume that the points β count μ times and that the points γ count λ times among the intersections of UHK, and to have written that formula provisionally $c(n-2) = 2\sigma + \mu\beta + \lambda\gamma$, when, proceeding as above, it would have been found that the formula could not be satisfied unless $\lambda = 1$, $\mu = 4$.

It only remains to examine the second of the formulæ (A). We have just assigned the values of all the quantities involved in it except t'. Substituting then these values, we find that the number of triple tangent planes to a surface of the nth degree is given by the formula

$$6t' = n(n-2)(n^7 - 4n^6 + 7n^5 - 45n^4 + 114n^3 - 111n^2$$
$$+ 548n - 960),$$

which verifies, as it gives $t' = 45$ when $n = 3$.

613. It was proved (Art. 279) that the points of contact of those edges of the tangent cone which touch the surface in two distinct points lie on a certain surface of degree $(n-2)(n-3)$. Now when the tangent cone is, as before, a complex cone $a + 2b + 3c$, it is evident that among these double tangents will be included those common edges of the cones ab, which meet the curves a, b in distinct points; and, similarly for the other pairs of cones. If then we denote by $[ab]$ the

number of the apparent intersections of the curves a and b, that is to say, the number of points in which these curves seen from any point of space seem to intersect, though they do not actually do so, the following formulæ will contain an analysis of the intersections of a, b, c, with the surface of the degree $(n-2)(n-3)$:

$$a(n-2)(n-3) = 2\delta + 2[ab] + 3[ac],$$
$$b(n-2)(n-3) = 4k + [ab] + 3[bc],$$
$$c(n-2)(n-3) = 6h + [ac] + 2[bc].$$

Now the number of apparent intersections of two curves is at once deduced from that of their actual intersections. For if cones be described having a common vertex and standing on the two curves, their common edges must answer either to apparent or actual intersections. Hence,

$$^*[ab] = ab - 2\rho,\ [ac] = ac - 3\sigma,\ [bc] = bc - 3\beta - 2\gamma - i.$$

Substituting these values, we have

$$\left.\begin{aligned} a(n-2)(n-3) &= 2\delta + 2ab + 3ac - 4\rho - 9\sigma \\ b(n-2)(n-3) &= 4k + ab + 3bc - 9\beta - 6\gamma - 3i - 2\rho \\ c(n-2)(n-3) &= 6h + ac + 2bc - 6\beta - 4\gamma - 2i - 3\sigma \end{aligned}\right\} \ldots (B).$$

The first and third of these equations are satisfied identically if we substitute for β, γ, ρ, σ, &c., the values used in the last article, to which we are to add $2\delta' = n(n-2)(n^2-9)$, $i' = 0$, and the value of h' got from (Art. 608),

$$2h' = n(n-2)(16n^4 - 64n^3 + 80n^2 - 108n + 156).$$

The second equation enables us to determine k' by the equation

$$8k' = n(n-2)(n^{10} - 6n^9 + 16n^8 - 54n^7$$
$$+ 164n^6 - 288n^5 + 547n^4 - 1058n^3 + 1068n^2 - 1214n + 1464);$$

from this expression the rank of the developable, of which b' is the cuspidal edge, can be calculated by the formula

$$R' = b'^2 - b' - 2k' - 6t' - 3\gamma'.$$

Putting in the values already obtained for these quantities we find

$$R' = n(n-2)(n-3)(n^2 + 2n - 4).$$

* If the surface have a double curve, but no cuspidal, there will still be a determinate number i of cuspidal points on the double curve, and the above equation receives the modification $[ab] = ab - 2\rho - i$. In determining, however, the degree of the reciprocal surface the quantity $[ab]$ is eliminated.

This is then the rank of the developable formed by the planes which have double contact with the given surface.

614. From formulæ (A) and (B) we can calculate the diminution in the degree of the reciprocal caused by the singularities on the original surface enumerated in Art. 609. If the degree of a cone diminish from m to $m-l$, that of its reciprocal diminishes from $m(m-1)$ to $(m-l)(m-l-1)$; that is to say, is reduced by $l(2m-l-1)$. Now the tangent cone to a surface is in general of degree $n(n-1)$, and we have seen that when the surface has double and cuspidal curves this degree is reduced by $2b+3c$. There is a consequent diminution in the degree of the reciprocal surface

$$D=(2b+3c)(2n^2-2n-2b-3c-1).$$

But the existence of double and cuspidal curves on the surface causes also a diminution in the number of double and cuspidal edges in the tangent cone. From the diminution in the degree of the reciprocal surface just given must be subtracted twice the diminution of the number of double edges, and three times that of the cuspidal edges. Now, from formulæ (A), we have

$$\kappa=(a-b-c)(n-2)+6\beta+4\gamma+3t.$$

But, since if the surface had no multiple curves, the number of cuspidal edges on the tangent cone would be $(a+2b+3c)(n-2)$, the diminution of the number of cuspidal edges is

$$K=(3b+4c)(n-2)-6\beta-4\gamma-3t.$$

Again, from the first system of equations in the last article, we have

$$(a-2b-3c)(n-2)(n-3)=2\delta-8k-18h-12[bc],$$

and putting for $[bc]$ its value

$$2\delta=(a-2b-3c)(n-2)(n-3)$$
$$+8k+18h+12bc-36\beta-24\gamma-12i.$$

But if the surface had no multiple curves, 2δ would

$$=(a+2b+3c)(n-2)(n-3).$$

The diminution then in the number of double edges is given by the formula

$$2H=(4b+6c)(n-2)(n-3)-8k-18h-12bc+36\beta+24\gamma+12i.$$

Thus the entire diminution in the degree of the reciprocal $D - 3K - 2H$ is, when reduced,

$$n(7b+12c) - 4b^2 - 9c^2 - 8b - 15c$$
$$+ 8k + 18h - 18\beta - 12\gamma - 12i + 9t.$$

615. The formulæ (B), reduced by the formula

$$a + 2b + 3c = n(n-1),$$

become

$$\left.\begin{aligned} a(-4n+6) &= 2\delta - a^2 - 4\rho - 9\sigma \\ b(-4n+6) &= 4k - 2b^2 - 9\beta - 6\gamma - 3i - 2\rho \\ c(-4n+6) &= 6h - 3c^2 - 6\beta - 4\gamma - 2i - 3\sigma \end{aligned}\right\} \ldots (C).$$

To each of these formulæ we add now four times the corresponding formula (A); and we simplify the results by writing for $a^2 - a - 2\delta - 3\kappa$, the degree n' of the reciprocal surface, by giving R the same meaning as in Art. 613, and by writing for $c^2 - c - 2h - 3\beta$, the degree S of the developable generated by the curve c; we thus obtain the formulæ in the more convenient shape,

$$\left.\begin{aligned} n' - a &= \kappa - \sigma \\ 2R &= 2\rho - \beta - 3i \\ 3S + c &= 5\sigma + \beta - 2i \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (D).$$

From the first of equations (A) and (D) we may also obtain the equation

$$(n-1)\, a = n' + \rho + 3\sigma,$$

the truth of which may be seen from the consideration that a, the curve of simple contact from any one point, intersects the first polar of any other point, either in the n' points of contact of tangent planes passing through the line joining the two points, or else in ρ points where a meets b, or the σ points where it meets c, since every first polar passes through the curves b, c.

616. The effect of multiple curves in diminishing the degree of the reciprocal may be otherwise investigated. The points of contact of tangent planes, which can be drawn through a given line, are the intersections with the surface of the curve of degree $(n-1)^2$, which is the intersection of the first polars of any two points on the line. Now, let us first consider the case when the surface has only an ordinary double curve of degree b. The first polars of the two points

pass each through this curve, so that their intersection breaks up into this curve b and a complemental curve d. Now, in looking for the points of contact of tangent planes through the given line, in the first place, instead of taking the points where the complex curve $b+d$ meets the surface, we are only to take those in which d meets it, which causes a reduction bn in the degree of the reciprocal. But, further, we are not to take all the points in which d meets the surface: those in which it meets the curve b have to be rejected; they are in number $2b(n-2)-r$ (Art. 346) where r is the rank of the system b. Now, these points consist of the r points on the curve b, the tangents at which meet the line through which we are seeking to draw tangent planes to the given surface, and of $2b(n-2)-2r$ points at which the two polar surfaces touch. These last are cuspidal points on the double curve b; that is to say, points at which the two tangent planes coincide, and they count for three in the intersections of the curve d with the given surface, since the three surfaces touch at these points; while the r points being ordinary points on the double curve only count for two. The total reduction then is

$$nb+2r+3\{2b(n-2)-2r\}=b(7n-12)-4r,$$

which agrees with the preceding theory.

If the curve b, instead of being merely a double curve, were a multiple curve on the surface of order of multiplicity p, I have found for the reduction of the degree of the reciprocal (see *Transactions of the Royal Irish Academy*, XXIII. p. 485)

$$b(p-1)(3p+1)n-2bp(p^2-1)-p^2(p-1)r,$$

for the reduction in the number of cuspidal edges of the cone of simple contact

$$b\{3(p-1)^2n-p(p-1)(2p-1)\}-p(p-1)(p-2)r,$$

and for twice the reduction in the number of its double edges

$$2bp(p-1)n^2-b(p-1)(14p-8)n$$
$$+bp(p-1)(8p-2)-b^2p^2(p-1)^2+p(p-1)(4p-6)r.$$

[For example, consider a surface of degree mp degenerating into p surfaces of degree m meeting one another in a

common curve of degree m^2; then $r=2m^2(m-1)$, and the reduction in class is

$$m^2\{mp(p-1)(3p+1)-2p(p^2-1)-2p^2(p-1)(m-1)\}$$
$$=mp(mp-1)^2-p\,.\,m(m-1)^3,$$

as it ought to be.]

The method of this article is not applied to the case where the surface has a cuspidal curve in the Memoir from which I cite, and I have not since attempted to repair the omission.

617. The theory just explained ought to enable us to account for the fact that the degree of the reciprocal of a developable reduces to nothing. This application of the theory both verifies the theory itself and enables us to determine some singularities of developables not given, Arts. 325, &c. We use the notation of the section referred to. The tangent cone to a developable consists of n planes; it has therefore no cuspidal edges and $\frac{1}{2}n(n-1)$ double edges. The simple curve of contact a consists of n lines of the system each of which meets the cuspidal edge m once, and the double curve x in $r-4$ points (see Art. 330). The curves m and x intersect at the a points of contact of the stationary planes of the system; for since there three consecutive lines of the system are in the same plane, the intersection of the first and third gives a point on the curve x. It is only on account of their occurrence in this example that I was led to include the points i in the theory.

We have then the following table:

Notation of this Chap., n, a, b, c; ρ, $\sigma, \kappa, \beta, h, i$; n', S;
Notation of Chap. XII., r, n, x, m; $n(r-4), n, 0, \beta, h, a$; $0, r$;

and the quantities t, γ, R, remain to be determined. On substituting these values in formulæ (A) and (D), Arts. 610, 615, we get the system of equations

$$\left.\begin{aligned} n(r-2)&=n(r-4+2),\\ x(r-2)&=n(r-4)+2\beta+3\gamma+3t,\\ m(r-2)&=2n+4\beta+\gamma,\\ -n&=-n,\\ 2R&=2n(r-4)-\beta-3a,\\ 3r+m&=5n-2a+\beta.\end{aligned}\right\}\ \ldots\ldots\ldots(E).$$

The first and fourth of these equations are identically true, and the sixth is verified by the equations of Arts. 326, 327. The three remaining equations determine the three quantities, whose values have not before been given, viz. t the number of "points on three lines" of the system; γ the number of points of the system through each of which passes another non-consecutive line of the system; and R the rank of the developable of which x is the cuspidal edge. These quantities being determined, we can by an interchange of letters write down the reciprocal singularities, the number of "planes through three lines," &c.

Ex. 1. Let it be required to apply the preceding theory to the case considered, Art. 329. Call k_1 the number of apparent double points on b, Art. 609, &c.

Ans. $\gamma = 6(k-3)(k-4)$, $\quad 3t = 4(k-3)(k-4)(k-5)$,

$$k_1 = (k-3)(2k^3 - 18k^2 + 57k - 65), \qquad R = 2(k-1)(k-3).$$

And for the reciprocal singularities

$\gamma' = 2(k-2)(k-3)$, $\quad 3t' = 4(k-2)(k-3)(k-4)$,

$$k_1' = (k-2)(k-3)(2k^2 - 10k + 11), \qquad R' = 6(k-3)^2.$$

Ex. 2. Two surfaces intersect the sum of whose degrees is p and their product q.

Ans. $\gamma = q(pq - 2q - 6p + 16)$.

This follows from the table, Art. 342, but can be proved directly by the method used (Arts. 343, 471), see *Transactions of the Royal Irish Academy*, XXIII. p. 469.

$$R = 3q(p-2)\{q(p-3) - 1\}.$$

Ex. 3. To find the singularities of the developable generated by a line resting twice on a given curve. The planes of this system are evidently "planes through two lines" of the original system: the class of the system is therefore y; and the other singularities are the reciprocals of those of the system whose cuspidal edge is x, calculated in this article. Thus the rank of the system, or the degree of the developable, is given by the formula

$$2R' = 2m(r-4) - \alpha - 3\beta.$$

618. Since the degree of the reciprocal of a ruled surface reduces always to the degree of the original surface (Art. 124) the theory of reciprocal surfaces ought to account for this reduction. I have not obtained this explanation for ruled surfaces in general, but some particular cases are examined and accounted for in the Memoir in the *Transactions of the Royal Irish Academy* already cited. I give only one example here.

Let the equation of the surface be derived, as in Art. 464, from the elimination of t between the equations

$$at^k + bt^{k-1} + \&c. = 0,\ a't^l + b't^{l-1} + \&c. = 0,$$

where a, a', &c., are any linear functions of the coordinates. Then if we write $k + l = \mu$, the degree of the surface is μ, having a double curve of degree $\frac{1}{2}(\mu - 1)(\mu - 2)$, on which are $\frac{1}{6}(\mu - 2)(\mu - 3)(\mu - 4)$ triple points. For the apparent double points of this double curve we have

$$2k = \tfrac{1}{4}(\mu - 2)(\mu - 3)(\mu^2 - 5\mu + 8);$$

and the developable generated by that curve is of degree $2(\mu - 2)(\mu - 3)$. It will be found then that we have $a = 2(\mu - 1)$, $b = \frac{1}{2}(\mu - 1)(\mu - 2)$, $\kappa = 3(\mu - 2)$, $\delta = 2(\mu - 2)(\mu - 3)$, values which agree with what was proved (Art. 614), that the number of cuspidal edges in the tangent cone is diminished by $3b(\mu - 2) - 3t$, while the double edges are diminished by $2b(\mu - 2)(\mu - 3) - 4k$. In verifying the separate formulæ (B) the remark, note, Art. 613, must be attended to.

I have also tried to apply this theory to the surface, which is the envelope of the plane $a\alpha^n + b\beta^n + c\gamma^n + \&c.$, where α, β, γ are arbitrary parameters, but have only succeeded when $n = 3$. We have here (see Art. 523, Ex. 2) $n = 12$, $n' = 9$, $a = 18$; b, being the number of cubics with two double points (that is, of systems of conic and line) which can be drawn through seven points, is 21; c is 24, since the cuspidal curve is the intersection of the surfaces of the fourth and sixth degree represented by the two invariants of the given cubic equation; for the same reason $h = 180$ and $S = c^2 - c - 2h - 3\beta = 192 - 3\beta$; t, being the number of cubics with three double points (that is, of systems of three right lines) which can be drawn through six points, is 15. The reciprocal of envelopes of the kind we are considering can have no cuspidal curve. This consideration gives $\kappa = 27$, $\delta = 108$. The formulæ (A) and (D) then give

$$180 = 27 + \rho + 2\sigma,\quad 210 = \rho + 2\beta + 3\gamma + 45,\quad 240 = 2\sigma + 4\beta + \gamma,$$
$$9 - 18 = 27 - \sigma,\quad 2R = 2\rho - \beta,\quad 3(192 - 3\beta) + 24 = 5\sigma + \beta.$$

These six equations determine the five unknowns and give one equation of verification. We have

$$\rho = 81,\ \sigma = 36,\ \beta = 42,\ \gamma = 0,\ R = 60.$$

[As a simpler example, consider the ruled quartic surface generated by a straight line resting on two given straight lines and on a given conic. Each of the given lines is double on the surface, and there is also one double generator, joining the traces of the two given lines on the plane of the conic. Thus the double curve is of degree 3, with one apparent double point, equivalent to a twisted cubic of rank 4. We must put $b=3$, $r=4$, $n=4$ in the first formula of Art. 616; the reduction in class is therefore

$$3(7 \times 4 - 12) - 4 \times 4 = 32 = 4 \times 3^2 - 4,$$

as it ought to be.

This reduction is twice that due to one of the non-intersecting double lines taken alone, which would give $b=1$, $r=0$, reduction $= 7 \times 4 - 12 = 16$; and we may consider that the double generator, meeting both the other double straight lines, causes no further reduction in class. For this double generator reduces the degree of the tangent cone by 2, but also reduces the number of its cuspidal edges by 6, on account of the two points of intersection of the generator with the second polar, each of which has been shown to decrease the number of cuspidal edges by 3. Since the cone has no stationary planes, $i=0$, and Plücker's equation

$$i - \kappa = 3(\nu - \mu)$$

shows that to decrease κ by 3 and μ by 1 has no effect upon the class.

More generally, consider the ruled surface of degree 2μ generated by a straight line resting on two given straight lines and on a given curve of degree μ. Each of the given lines is μ-fold on the surface, and for the reduction in class we must put $p = \mu$, $b = 1$, $r = 0$, $n = 2\mu$ in the second formula of Art. 616, giving

$$2\{(\mu-1)(3\mu+1)2\mu - 2\mu(\mu^2-1)\} = 8\mu^2(\mu-1) = 2\mu(2\mu-1)^2 - 2\mu,$$

as it ought to be. There are also a certain number of double generators, each resting twice upon the given curve, but as before each meets the second polar in two points not on the double lines, and has no effect upon the reduction in class.]

619. It may be mentioned here that the Hessian of a ruled surface meets the surface only in its multiple curves, and in the generators each of which is intersected by one consecutive. For (Art. 463) if xy be any generator, that part of the equation which is only of the first degree in x and y is of the form $(xz+yw)\,\phi$. Then (Art. 287) the part of the Hessian which does not contain x and y is

$$\left\{\left(\phi + z\frac{d\phi}{dz}\right)\left(\phi + w\frac{d\phi}{dw}\right) - wz\frac{d\phi}{dz}\,\frac{d\phi}{dw}\right\}^2,$$

which reduces to ϕ^4. But xy intersects ϕ only in the points where it meets multiple curves. But if the equation be of the form $ux + vy^2$ (Art. 287) the Hessian passes through xy. Thus in the case considered in the last article, the number of lines which meet one consecutive are easily seen to be $2\,(\mu-2)$; and the curve UH, whose degree is $4\mu\,(\mu-2)$, consists of these lines, each counting for two and therefore equivalent to $4\,(\mu-2)$ in the intersection, together with the double curve equivalent to $4\,(\mu-1)\,(\mu-2)$. Again, if a surface have a multiple curve whose degree is m, and order of multiplicity p, it will be a curve of degree $4\,(p-1)$ on the Hessian, and will be equivalent to $4\,mp\,(p-1)$ on the curve UH. Now the ruled surface generated by a line resting on two right lines and on a curve m (which is supposed to have no actual multiple point) is of degree $2m$, having the right lines as multiples of order m, having $\frac{1}{2}m\,(m-1)+h$ double generators, and $2r$ generators which meet a consecutive one. Comparing then the degree of the curve UH with the sum of the degrees of the curves of which it is made up, we have

$$16m\,(m-1) = 8m\,(m-1) + 4m\,(m-1) + 8h + 4r,$$

an equation which is identically true.

ADDITION ON THE THEORY OF RECIPROCAL SURFACES.

620. [The third and fourth editions of this work contained a memoir by Cayley on reciprocal surfaces, which is reprinted in his *Collected Mathematical Papers* (XI. p. 225, VI. p. 582). This was left in an imperfect state (Cayley, *C.M.P.* VI.

p. 595; Wölffling, *Math.-naturwiss. Mitteilungen Württ.* I. p. 22 (1899), II. p. 87, III. p. 55). The following articles, while retaining Cayley's notation and arrangement, are based on Zeuthen's memoir, "Révision et extension des formules numériques de la théorie des surfaces réciproques," *Math. Ann.* x. p. 446 (1876).]

It will be convenient to give the following complete list of the quantities which present themselves. The definitions in the first and second columns have the properties which are the most general when the surface is regarded as a locus and as an envelope respectively. The last set have the same generality from either point of view.

NUMBERS NOT REFERRING TO SINGULARITIES.

Section by any plane.	*Tangent cone from any point.*
n degree	n' class
a' class	a degree
δ' double tangents	δ double edges
κ' inflexions	κ cuspidal edges

ORDINARY SINGULARITIES.

Node-couple torse.	*Double curve.*
b' class	b degree
q' degree	q class
k' apparent double planes	k apparent double points
t' triple planes	t triple points
ρ' degree of curve of contact	ρ class of nodal torse
Spinode torse.	*Cuspidal curve.*
c' class	c degree
r' degree	r class
h' apparent double planes	h apparent double points
σ' degree of curve of contact	σ class of cuspidal torse
Common planes of the two torses.	*Intersections of the two curves.*
β' stationary on spinode torse	β cuspidal on cuspidal curve
γ' stationary on node-couple torse	γ cuspidal on double curve

ORDINARY SINGULARITIES OF A SURFACE ALREADY POSSESSING:

a double curve.	*a node-couple torse.*
j pinch-points	j' pinch-planes
a cuspidal curve.	*a spinode torse.*
χ close-points	χ' close-planes

EXTRAORDINARY SINGULARITIES.

Special multiple points.	*Special singular planes.*
B binodes	B' bitropes
U unodes	U' unitropes
O osculatory planes	O' osculatory points
Tangent cone at a general multiple point.	*Curve of contact of a general singular tangent plane.*
μ degree	μ' class
ν class	ν' degree
$\eta+y$ double edges, of which y touch the double curve	$\eta'+y'$ double tangents, of which y' lie on the node-couple torse
$\zeta+z$ cuspidal edges, of which z touch the cuspidal curve	$\zeta'+z'$ stationary tangents, of which z' lie on the spinode torse
u double planes	u' nodes
v stationary planes	v' cusps

we shall also write

x for $\nu+2\eta+3\zeta$	x' for $\nu'+2\eta'+3\zeta'$
(Σ denotes a sum extended to these multiple points.)	(Σ' denotes a sum extended to these singular planes.)

Tacnodes.

f nodes on double curve
d cusps on double curve
g nodes on cuspidal curve
e cusps on cuspidal curve
i intersections of the two curves.

Zeuthen has shown (*Math. Ann.*, IX. p. 321) that at these tacnodes the tangent planes have the reciprocal properties, so that the same numbers are given by the reciprocal definitions.

621. We evidently have

$$a' = a.$$

The definitions of ρ and σ agree with those given, Art. 609: the nodal torse is the torse enveloped by the tangent planes along the double curve; if the double curve meets the curve of contact a at an ordinary point, then a tangent plane of the nodal torse passes through the arbitrary point, that is, ρ will be the number of these planes which pass through the arbitrary point, viz. the class of the torse. So also the cuspidal torse is the torse enveloped by the tangent planes along the cuspidal curve; and σ will be the number of these tangent planes which pass through the arbitrary point, viz. it will be the class of the torse. Again, as regards ρ' and σ': the node-couple torse is the envelope of the bitangent planes of the surface, and the node-couple curve is the locus of the points of contact of these planes; similarly, the spinode torse is the envelope of the parabolic planes of the surface, and the spinode curve is the locus of the points of contact of these planes; viz. it is the curve UH of simple intersection of the surface and its Hessian; the two curves are the reciprocals of the nodal and cuspidal torses respectively, and the definitions of ρ', σ' correspond to those of ρ and σ.

622. The j pinch-points are not singular points of the double curve *per se*, but are singular in regard to the curve as double curve of the surface; viz. a pinch-point is a point at which the two tangent planes are coincident. The section of the surface by a plane passing through a pinch-point has a cusp, and the curves of contact a of all tangent cones pass through the point and have the same tangent there. The j' pinch-planes have the reciprocal properties, and touch the surface along straight lines.

623. The χ close-points are points of the cuspidal curve at which the section of the surface has a tacnode instead of a cusp. In the neighbourhood of a close-point, the surface

resembles a flattened cone, limited by the cuspidal curve and the contour of the surface. The curves of contact a of all tangent cones pass through the point and have the same tangent there. The χ' close-planes, like the pinch-planes, touch the surface along straight lines.

624. The γ' planes are stationary on the node-couple torse, and cut the surface in sections having a node and a cusp (which are not multiple points of the surface). It follows that the points γ are points where the cuspidal curve with the two sheets (or say rather half-sheets) belonging to it are intersected by another sheet of the surface; the curve of intersection with such other sheet belonging to the double curve of the surface has evidently a stationary (cuspidal) point at the point of intersection.

The β' planes are stationary on the spinode torse, and cut the surface in sections having a tacnode (which is not a multiple point of the surface) instead of a cusp. The tangent at one of these points to the section is a generator both of the node-couple torse and of the spinode torse. Similarly, the tangent to the cuspidal curve at one of its β cusps is also the tangent at that point to the double curve; then intersecting the two curves by a series of parallel planes, any plane which is, say, above the cusp, meets the cuspidal curve in two real points and the double curve in one real point, and the section of the surface is a curve with two real cusps and a real node; as the plane approaches the cusp, these approach together, and, when the plane passes through the cusp, unite into a singular point in the nature of a triple point (= node + two cusps); and when the plane passes below the cusp, the two cusps of the section become imaginary, and the double curve changes from crunodal to acnodal.

625. At a point i the double curve crosses the cuspidal curve, being on the side away from the two half-sheets of the surface acnodal, and on the side of the two half-sheets crunodal, viz. the two half-sheets intersect each other along this

portion of the double curve. There is at the point a single tangent plane, which is a plane i'.

The equivalent numbers of double points and cusps on the double curve are

$$\bar{f} = f + 3t + 3O' + \Sigma\left[\tfrac{1}{2}y\,(y-1)\right] + \Sigma'\left[2\zeta'(\nu'-3) + \tfrac{3}{2}\eta'(\eta'-1) + \eta'\zeta' + 3\zeta'(\zeta'-1) + u'\right]$$

and

$$\bar{d} = d + \gamma + \Sigma'\left[\eta'(\nu'-4) + 2\eta'\zeta'\right],$$

and on the cuspidal curve,

$$\bar{g} = g + 6\chi' + 12B' + U' + 4O' + \Sigma\left[\tfrac{1}{2}z\,(z-1)\right] + \Sigma'\zeta'$$

and

$$\bar{e} = e + \beta + 2O'.$$

The equivalent number of intersections of the two curves is

$$\bar{i} = i + 3\beta + 2\gamma + 12O' + \Sigma\,[yz] + \Sigma'\,[4\eta' + 4\zeta' + v'].$$

(Zeuthen's symbols h, k stand for $h + \bar{g}$, $k + \bar{f}$ respectively).

626. The osculatory point O' is understood rather more easily by means of the reciprocal singularity of the osculatory plane O; this is a tangent plane meeting the surface in a curve having the point of contact for a triple point.

[Cayley also considered C isolated cnicnodes, for which $\mu = \nu = 2$, $y = \eta = z = \zeta = u = v = 0$; and ω off-points, or triple points lying on the cuspidal curve, for which $\mu = \nu = 3$, $z = v = 1$, $y = \eta = \zeta = u = 0$; with the reciprocal singularities. All these sets are included in Σ and Σ'. It is assumed that if at a general multiple point the tangent cone breaks up, then among its parts there are no planes, and no repeated sheets.

A full description of all the singularities is given in Zeuthen's memoir cited in Art. 620.]

627. These quantities satisfy the following equations:

(1) $\quad n\,(n-1) = a + 2b + 3c$

(2) $\quad a\,(a-1) = n + 2\delta' + 3\kappa'$

(3) $\quad c - \kappa' = 3\,(n-a)$

(4) $\quad a\,(n-2) = \kappa + \rho + 2\sigma - B + \Sigma\,[x\,(\mu-2) - \eta - 2\zeta]$

(5) $$b(n-2) = 3t + \rho + 2\beta + 3\gamma + 9O' + \Sigma\,[y\,(\mu-2)]$$

(6) $$c(n-2) = 2\sigma + 4\beta + \gamma + 8\chi' + 16B' + 12O' + \Sigma\,[z\,(\mu-2)]$$

(7) $$a(n-2)(n-3) = 2\delta - 6U + 3(ac - 3\sigma - \chi) + 2(ab - 2\rho - j) + \Sigma\,[x(-4\mu+7) + 2\eta + 4\zeta]$$

(8) $$b(n-2)(n-3) = 4k + 9O' + (ab - 2\rho - j) + 3(bc - i) + \Sigma\,[y\,(\mu-2)(\mu-3) - xy]$$

(9) $$c(n-2)(n-3) = 6h + 18O' + (ac - 3\sigma - \chi) + 2(bc - i) + \Sigma\,[z\,(\mu-2)(\mu-3) - xz]$$

(10) $$b(b-1) = q + 2(k + \bar{f}) + 3\bar{d}$$

(11) $$c(c-1) = r + 2(h + \bar{g}) + 3\bar{e}$$

and eleven other equations formed from these by exchanging accented and unaccented letters (except a, f, d, g, e, i), together with one other independent relation.

628. This new relation may be presented under several different forms, equivalent to one another in virtue of the foregoing twenty-two relations; one of these is

$$\sigma + 2r - 3c - 4j' - 3\chi' + 2O' - 14U' - \Sigma'\,[2\mu' + x' + 6\eta' + 8\zeta']$$

= the same expression with accented and unaccented letters interchanged.

Another form expresses that the surface and its reciprocal have the same deficiency p, for which Zeuthen gives the equation

$$\begin{aligned}24(p+1) = 24n - 12a - 15c + c' + 3r + 6g + 9e + 2\sigma + \beta \\ + 6\chi + 12\chi' + 8B + 24B' + 18U + 6U' + 6O' \\ + \Sigma\,[3x + 3z + 2\eta + 4\zeta] \quad + 6\Sigma'\zeta'.\end{aligned}$$

629. From the equations of Art. 627 we deduce

$$\begin{aligned}n' = n(n-1)^2 - n(7b + 12c) + 4b^2 + 8b + 9c^2 + 15c - 8k - 18h - 9t \\ + 18\beta + 12\gamma + 12i - 24\chi' - 3B - 48B' - 6U + 9O' \\ + \Sigma\,[2xy + 3xz + 12yz - x(\mu-1) - 3(y+z)(\mu-2) \\ - (2y+3z)(\mu-2)(\mu-3) - (\eta + 2\zeta)] \\ + 12\Sigma'\,[4\eta' + 4\zeta' + v'],\end{aligned}$$

which shows the effect of each singularity in reducing the class.

630. In the case of the general surface of degree n without singularities, b, q, k, f, d, t, ρ, j; c, r, h, g, e, σ, χ; β, γ, i; χ', B, B', U, U', O, O'; Σ, Σ' all vanish and we have:

$$
\begin{aligned}
n &= n,\\
a &= n(n-1),\\
\delta &= \tfrac{1}{2}n(n-1)(n-2)(n-3),\\
\kappa &= n(n-1)(n-2),\\
n' &= n(n-1)^2,\\
a' &= n(n-1),\\
\delta' &= \tfrac{1}{2}n(n-2)(n^2-9),\\
\kappa' &= 3n(n-2),\\
b' &= \tfrac{1}{2}n(n-1)(n-2)(n^3-n^2+n-12),\\
k' &= \tfrac{1}{8}n(n-2)(n^{10}-6n^9+16n^8-54n^7+164n^6-288n^5\\
&\qquad +547n^4-1058n^3+1068n^2-1214n+1464),\\
t' &= \tfrac{1}{6}n(n-2)(n^7-4n^6+7n^5-45n^4\\
&\qquad +114n^3-111n^2+548n-960),\\
q' &= n(n-2)(n-3)(n^2+2n-4),\\
\rho' &= n(n-2)(n^3-n^2+n-12),\\
c' &= 4n(n-1)(n-2),\\
h' &= \tfrac{1}{2}n(n-2)(16n^4-64n^3+80n^2-108n+156),\\
r' &= 2n(n-2)(3n-4),\\
\sigma' &= 4n(n-2),\\
\beta' &= 2n(n-2)(11n-24),\\
\gamma' &= 4n(n-2)(n-3)(n^3-3n+16).
\end{aligned}
$$

THE END.

INDEX OF SUBJECTS. VOLUME II.

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